

# A New Approach to Incorporate Uncertainty into Euler's Method

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## **Abstract**

Fuzzy set provides a powerful technique to introduce uncertainty into numerical methods. However, the computations of fuzzy sets often face difficult problems. This is due to non-applicability of common existing methods, severe overestimation in computation, or very high computational complexity. This paper proposes a new strategy to introduce uncertainty into Euler's method. It consists of two parts. First, we propose a new fuzzy version of Euler's method, which takes into account the dependency problem that arises in the classical Euler's method. Second, we perform optimisation technique to approximate the solution of differential equations with fuzzy initial values. This combination turns out to be a great tool to tackle uncertainty in any numerical method. One example is provided to show the capability of our proposed methods compared to the conventional fuzzy version of Euler's method proposed in the literature.

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**Keywords:** Euler's Method, Fuzzy Initial Value, Fuzzy Set, Optimisation

# 1 Introduction

In modelling real physical phenomena, differential equations play a significant role in science and engineering. They often represent an idealisation of the real physical phenomena involved. The real physical phenomena, however, are pervaded with uncertainty. The uncertainty can arise in the experimental part, the data collection, the measurement process, as well as when determining the initial values. These are patently obvious when dealing with “living” materials, such as soil, water, microbial populations, etc. Various theories exist for describing this uncertainty and the most popular one being fuzzy set theory [1].

When a real physical phenomenon is transform into a deterministic initial value problem, namely

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad (1)$$

we cannot usually be sure that the model is perfect. For example, the initial value in (1) may not be known exactly. If this is the case, then it would be natural to study differential equations with fuzzy initial values. For the initiation of this aspect, the necessary calculus has been investigated (see [2, 3, 4, 5, 6, 7, 8, 9, 10]).

In general, the differential equations with fuzzy initial values do not always have solutions which we can obtain using analytical methods. In fact, many of real physical phenomena encountered, are almost impossible to solve by this technique. Due to this, some authors have proposed numerical methods to approximate the solutions of differential equations with fuzzy initial values. One of the earlier contributions was the fuzzy Euler’s method proposed by Ma *et al.* [11]. In [12], the authors have implemented this method to explore hybrid fuzzy systems. Unfortunately, the proposed method only works in practice for some differential equations with fuzzy initial values and the computational aspects are trivial.

Moreover, the authors do not take into account the dependency problem that exists in fuzzy setting. This is frequently the case in fuzzy computations. In [13], the authors have developed the 4-th order Runge-Kutta method for solving differential equations with fuzzy initial values. However, their works share the same problems as in [11]. We can see the same problems in another papers written by Khastan and Ivaz [14], Palligkinis *et al.* [15], Pederson and Sambandham [16], Abbasbandy and Allahviranloo [17] and Duraisamy and Usha [18]. By taking into account the dependency problem and the computational approach proposed by Ahmad *et al.* [19], we develop a new fuzzy version of

Euler's method for a more general class of problems.

This paper is organised as follows: in Section 2, we recall some basic definitions and theoretical background we need throughout this paper. The theory of differential equations with fuzzy initial values is presented in Section 3. In Section 4, we propose a new fuzzy version of Euler's method for solving differential equations with fuzzy initial values. In Section 5, we give a numerical example to approximate the solution of non-linear differential equation with fuzzy initial value. We conclude with an overview of the benefits of the proposed method in Section 6.

## 2 Preliminaries

In this section, the basic idea of fuzzy sets will be introduced and some important concepts will be explained.

### 2.1 Fuzzy sets

The notion of a fuzzy set is an extension of that of a classical or crisp set. Let  $X$  be a set of objects, called the universe, whose generic elements are denoted by  $x$ . Membership in a subset  $A$  of  $X$  can be viewed as a characteristic function, or membership function  $A : X \rightarrow \{0, 1\}$  such that

$$A(x) = \begin{cases} 1 & , \text{if } x \in A, \\ 0 & , \text{if } x \notin A. \end{cases} \quad (2)$$

The set  $\{0, 1\}$  is called the valuation set. If the valuation set is allowed to be the real unit interval  $[0, 1]$ , then  $A$  is called a fuzzy subset of  $X$  or simply a fuzzy set in  $X$ . In this case,  $A(x)$  is interpreted as the degree of membership of the element  $x$  in the fuzzy set  $A$ .

**Definition 1** *Let  $A$  be a fuzzy set defined on  $\mathbb{R}$ . The support of  $A$  is a crisp set of all points on  $\mathbb{R}$  such that the membership degree of  $A$  is non-zero, that is*

$$\text{supp}(A) = \{x \in \mathbb{R} \mid A(x) > 0\} .$$

**Definition 2** *Let  $A$  be a fuzzy set defined on  $\mathbb{R}$ . The core of  $A$  is the crisp set of all points on  $\mathbb{R}$  such that the membership degree of  $A$  is 1, that is*

$$\text{core}(A) = \{x \in \mathbb{R} \mid A(x) = 1\} .$$

**Definition 3** *Let  $A$  be a fuzzy set on  $\mathbb{R}$ .  $A$  is called a fuzzy interval if:*

(i) *A is normal: there exists  $x_0 \in \mathbb{R}$  such that  $A(x_0) = 1$ ;*

(ii) *A is convex: for all  $x, y \in \mathbb{R}$  and  $0 \leq \lambda \leq 1$ , it holds that*

$$A(\lambda x + (1 - \lambda)y) \geq \min(A(x), A(y));$$

(iii) *A is upper semi-continuous: for any  $x_0 \in \mathbb{R}$ , it holds that*

$$A(x_0) \geq \lim_{x \rightarrow x_0^\pm} A(x);$$

(iv)  $[A]^0 = \overline{\{x \in \mathbb{R} \mid A(x) > 0\}}$  *is a compact subset of  $\mathbb{R}$ .*

The  $\alpha$ -cut of a fuzzy interval  $A$ , with  $0 < \alpha \leq 1$  is the crisp set

$$[A]^\alpha = \{x \in \mathbb{R} \mid A(x) \geq \alpha\}.$$

For a fuzzy interval  $A$ , its  $\alpha$ -cuts are closed intervals in  $\mathbb{R}$ ; we denote them by

$$[A]^\alpha = [a_1^\alpha, a_2^\alpha].$$

**Definition 4** *A fuzzy interval  $A$  is called a triangular fuzzy interval if its membership function has the following form:*

$$A(x) = \begin{cases} 0 & , \text{if } x < a, \\ \frac{x-a}{b-a} & , \text{if } a \leq x \leq b, \\ \frac{c-x}{c-b} & , \text{if } b \leq x \leq c, \\ 0 & , \text{if } x > c, \end{cases}$$

*and its  $\alpha$ -cuts are simply  $[A]^\alpha = [a + \alpha(b - a), c - \alpha(c - b)]$ ,  $\alpha \in (0, 1]$ .*

In this paper, the set of all fuzzy intervals is denoted by  $\mathcal{F}(\mathbb{R})$ .

## 2.2 The extension principle

Any crisp function can be extended to take fuzzy set as arguments by applying Zadeh's extension principle [1]. Let  $f$  be a function from  $X$  to  $Y$ . Given a fuzzy set  $A$  in  $X$ , we want to find a fuzzy set  $B = f(A)$  in  $Y$  that is induced by  $f$ . If  $f$  is a strictly monotone function then we can extend  $f$  to fuzzy set as follow:

$$f(A)(y) = \begin{cases} A(f^{-1}(y)) & , \text{if } y \in \text{range}(f), \\ 0 & , \text{if } y \notin \text{range}(f). \end{cases} \quad (3)$$

It is clear that (3) can be easily calculated by determining the membership at the end points of the  $\alpha$ -cuts of  $A$ . However, in general, the process of finding the fuzzy set  $B = f(A)$  is more complicated and cannot be gathered easily. For example, if  $f$  is a non-monotone function, then the problem can arise when two or more distinct points in  $X$  are mapped to the same point in  $Y$ . If this is the case, then the above equation may take two or more different values. This requires a new extension of (3) as shown below:

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x) & , \text{ if } y \in \text{range}(f), \\ 0 & , \text{ if } y \notin \text{range}(f), \end{cases} \tag{4}$$

where

$$f^{-1}(\{y\}) = \{x \in X \mid f(x) = \{y\}\} .$$

The equation (4) is called Zadeh's extension principle [1].

In [20], the authors have shown that if  $f : X \rightarrow Y$  is a continuous function, then  $f : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$  is a well-defined function, and

$$[f(A)]^\alpha = f([A^\alpha]),$$

for all  $\alpha \in [0, 1]$  and  $A \in \mathcal{F}(X)$ .

### 3 Fuzzy Initial Value Problems

In this section, we first consider the following ordinary differential equation:

$$\begin{cases} x'(t) = f(t, x(t)), & t \in [t_0, T] \\ x(t_0) = x_0, \end{cases} \tag{5}$$

where  $f : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function defined on  $[t_0, T]$  with  $T > 0$  and  $x_0 \in \mathbb{R}$ . Suppose that the initial condition in (5) is uncertain and modelled by a fuzzy interval, then we have the following fuzzy initial value problem [10]:

$$\begin{cases} x'(t) = f(t, x(t)), & t \in [t_0, T] \\ x(t_0) = X_0, \end{cases} \tag{6}$$

where  $f : [t_0, T] \times \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})$  is continuous function defined on  $[t_0, T]$  with  $T > 0$  and  $X_0 \in \mathcal{F}(\mathbb{R})$  with  $\alpha$ -cuts denoted by  $[X_0]^\alpha = [x_{0,1}^\alpha, x_{0,2}^\alpha]$  for  $\alpha \in (0, 1]$ . If  $X$  is a fuzzy interval, then from Zadeh's extension principle [1] we have

$$f(t, X)(z) = \begin{cases} \sup_{z=f(t,s)} X(s), & \text{ if } z \in \text{range}(f), \\ 0, & \text{ if } z \notin \text{range}(f). \end{cases}$$

It follows that

$$[f(t, X)]^\alpha = [\min \{f(t, u) \mid u \in [x_1^\alpha, x_2^\alpha]\}, \\ \max \{f(t, u) \mid u \in [x_1^\alpha, x_2^\alpha]\}].$$

This leads to the following lemma:

**Lemma 1** [21] *Let  $f : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function defined on  $[t_0, T]$  with  $T > 0$ . If  $X$  is a fuzzy interval defined on  $\mathbb{R}$ , then we have*

$$[f(t, X)]^\alpha = f(t, [X]^\alpha),$$

for all  $\alpha \in [0, 1]$ .

Let  $X : [t_0, T] \rightarrow \mathcal{F}(\mathbb{R})$  be a fuzzy process which derivative defined by [10]

$$[X'(t)]^\alpha = [x_1^{\alpha'}(t), x_2^{\alpha'}(t)], \quad \alpha \in (0, 1].$$

If the derivative of the fuzzy process exists and satisfies the following conditions:

$$x_1^{\alpha'}(t) = \min \{f(t, u) \mid u \in [x_1^\alpha, x_2^\alpha]\}, \quad x_1^\alpha(t_0) = x_{0,1}^\alpha$$

$$x_2^{\alpha'}(t) = \max \{f(t, u) \mid u \in [x_1^\alpha, x_2^\alpha]\}, \quad x_2^\alpha(t_0) = x_{0,2}^\alpha$$

then, the fuzzy process is the solution of (6) on  $t \in [t_0, T]$  with  $T > 0$  and  $\alpha \in (0, 1]$ . If we solve (6) analytically, then we have to verify that the interval  $[X(t)]^\alpha = [x_1^\alpha(t), x_2^\alpha(t)]$  satisfies the following theorem:

**Theorem 1** [22] *If  $X : [t_0, T] \rightarrow \mathcal{F}(\mathbb{R})$  is a fuzzy solution, then*

- (i)  $[X(t)]^\alpha$  is nonempty compact subset of  $\mathbb{R}$ ;
- (ii)  $[X(t)]^{\alpha_2} \subseteq [X(t)]^{\alpha_1}$  for  $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ ; and
- (iii)  $[X(t)]^\alpha = \bigcap_{n=1}^{\infty} [X(t)]^{\alpha_n}$  for any nondecreasing sequence  $\alpha_n \rightarrow \alpha$  in  $[0, 1]$ .

## 4 The Proposed Method

In [11], the authors have proposed a fuzzy version of Euler's method to approximate the solution of differential equations with fuzzy initial values. However, the authors do not take into account the dependency problem when adding of two fuzzy intervals. This leads to overestimation in computation.

First, we recall the classical Euler's method

$$x_{i+1} = x_i + hf(t_i, x_i), \quad i = 0, 1, 2, \dots, N - 1. \tag{7}$$

Let us remark about dependency problem appears in (7). Since the arguments  $x_i$  are dependent, then we have to define the right hand side of (7) as a new function. We denote it by  $\Phi_h(t_i, x_i) = x_i + hf(t_i, x_i)$ . Hence, the right hand side of (7) turns to

$$x_{i+1} = \Phi_h(t_i, x_i), \quad i = 0, 1, 2, \dots, N - 1, \tag{8}$$

where  $\Phi_h : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function defined on  $[t_0, T]$  with  $T > 0$ . By applying Zadeh's extension principle to (8), then we have the following new fuzzy version of Euler's method:

$$X_{i+1} = \Phi_h(t_i, X_i), \tag{9}$$

where  $\Phi_h : [t_0, T] \times \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})$ . The membership function of  $\Phi_h(t_i, X_i)$  can be defined as follow:

$$\Phi_h(t_i, X_i)(z_i) = \begin{cases} \sup_{z_i = \Phi_h(t_i, s_i)} X_i(s_i), & \text{if } z_i \in \text{range}(\Phi_h), \\ 0, & \text{if } z_i \notin \text{range}(\Phi_h). \end{cases}$$

It follows that

$$[\Phi_h(t_i, X_i)]^\alpha = [\min \{ \Phi_h(t_i, u) | u \in [x_{1,i}^\alpha(t), x_{2,i}^\alpha(t)] \}, \max \{ \Phi_h(t_i, u) | u \in [x_{1,i}^\alpha(t), x_{2,i}^\alpha(t)] \}].$$

If  $[X_{i+1}]^\alpha = [x_{1,i+1}^\alpha, x_{2,i+1}^\alpha]$ , then we can express (9) in terms of  $\alpha$ -cuts as follows:

$$x_{1,i+1}^\alpha = \min \{ \Phi_h(t_i, u) | u \in [x_{1,i}^\alpha(t), x_{2,i}^\alpha(t)] \}, \tag{10}$$

$$x_{2,i+1}^\alpha = \max \{ \Phi_h(t_i, u) | u \in [x_{1,i}^\alpha(t), x_{2,i}^\alpha(t)] \}. \tag{11}$$

Our purpose here is to generate accurate approximations at each  $\alpha$ -cut. We begin by making a partition of the form  $t_0 < t_1 < t_2 < \dots < t_{N-1} < t_N = T$  on the interval  $[t_0, T]$ . This partition is uniformly spaced, that is the partition points are  $t_i = t_0 + ih, i = 0, 1, 2, \dots, N$  and the partition spacing  $h = \frac{T-t_0}{N} > 0$  is sufficiently small and we called it the step-size or step-length.

In this study, the computations of (10) and (11) will be performed by using the method proposed by Ahmad *et al.* [19]. The method is based on optimisation technique. One of the reasons of using this method is that it requires only few function evaluations at each partition point.

## 5 Numerical Example

In this section, we present a numerical example to show the capability of our proposed method compared to the conventional fuzzy version of Euler's method proposed in [11].

Consider the following non-linear differential equation with fuzzy initial value:

$$\begin{cases} x'(t) = t \cos(x), & t \in [0, 3] \\ x(0) = (\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2}). \end{cases} \quad (12)$$

Since the exact solution cannot be found analytically, we need a numerical method to approximate the solution of (12). First, we divide the interval  $[0, 3]$  into 20 uniformly spaced subintervals and proceed with the numerical method proposed in Section 4. The final results are shown in Fig. 1.

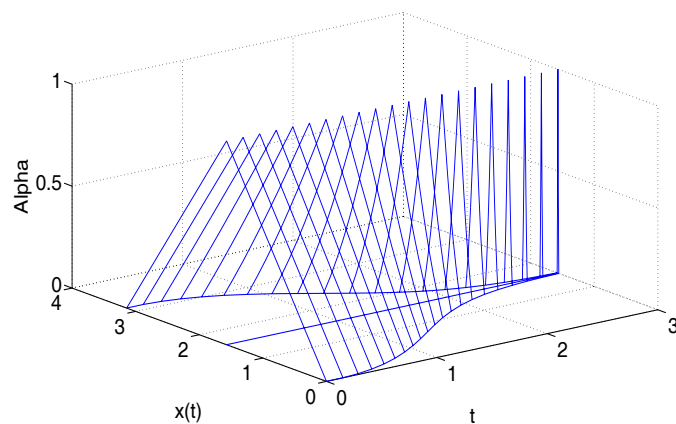


Figure 1: The approximation solution obtained by using the method proposed in this paper.

From the graph, we can see that the approximation solution has decreasing length of its support as  $t$  increases. In contrast, applying the numerical method proposed in [11], the approximation solution has increasing length of its support (see Fig. 2). This behaviour can be interpreted as the approximation solution becomes fuzzier and fuzzier as  $t$  increases. Hence, the approximation solution behaves quite differently from the crisp solution i.e. at several initial values.

As showed in Figs. 1 and 2, the differential equation has two contradict solutions. Fig. 1 shows converge fuzzy solution which resulted from our proposed



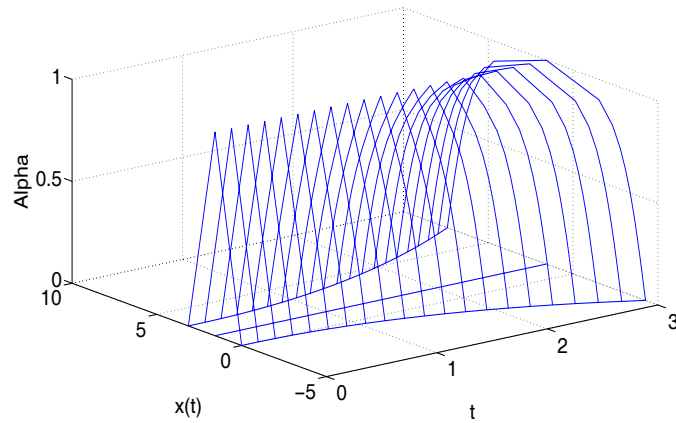


Figure 2: The approximation solution obtained by using the method proposed in [11].

method. While Fig. 2 shows diverge fuzzy solution which resulted from the numerical method proposed in [11]. Which approximation solution satisfies the differential equation? The only way to check this is by sketching a direction field, a way of predicting the qualitative behaviour of the solution of differential equation. The direction field represents the slope of approximation solution in the  $tx$ -plane. It is represented by the collection of narrow lines. From the practical point of view, if the approximation solution follows the direction field, then the approximation solution is the solution to the differential equation.

From the numerical results, we plot the direction field of (12). It is represented by narrow lines as showed in Fig. 3. If we look at the figure, the slope of the approximation solution obtained by our proposed method follows the direction field. This is enough to prove that our proposed method produced better solution. In contrary, the approximation solution obtained by using the method proposed in [11] does not follow the direction field. Moreover, it has overestimation in computation as  $t$  increases. This is always the case when we consider the same variable as independent in fuzzy interval computations (see Eq. (7)).

To quantify the effect of overestimation, we calculate the local degree of overestimation at a specific level of  $\alpha_j$  according to the following equation:

$$\widehat{\Psi}_i^{\alpha_j} = \frac{W[\widehat{\Phi}_h(t_i, X_i^{\alpha_j})] - W[\Phi_h(t_i, X_i^{\alpha_j})]}{W[\Phi_h(t_i, X_i^{\alpha_j})]}, \tag{13}$$

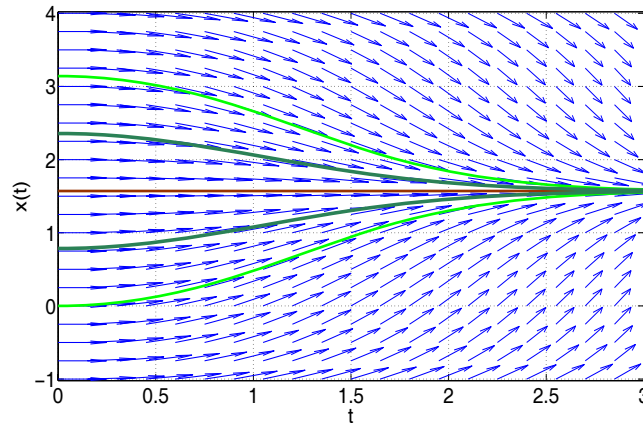


Figure 3: The narrow lines show the direction field of  $x'(t) = t \cos(x)$ . The solid curves denote the approximation solution of  $x'(t) = t \cos(x)$  with  $x(0) = (\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$  at  $\alpha = 0, 0.5, 1$  generated by the new fuzzy Euler’s method, using a step size of  $h = 0.15$ .

where  $W[\widehat{\Phi}_h(t_i, X_i^{\alpha_j})]$  is the width of approximation solution obtained in [11] at  $t_i$  with  $\alpha_j \in [0, 1)$  and  $W[\Phi_h(t_i, X_i^{\alpha_j})]$  is the width of approximation solution obtained by our proposed method at  $t_i$  with  $\alpha_j \in [0, 1)$ . The width,  $W$  of an interval is defined as follow:

$$W([a, b]) = b - a. \tag{14}$$

The local degrees of overestimation at  $t = 1.05$  for each  $\alpha$  level are given in Table 1.

Table 1: Local degree of overestimation at  $t = 1.05$

$\alpha$	$\widehat{\Phi}_h(t_{1.05}, X_{1.05}^\alpha)$	$\Phi_h(t_{1.05}, X_{1.05}^\alpha)$	$\widehat{\Psi}_i^\alpha(\%)$
0	[-0.4725, 3.6141]	[0.4613, 2.6803]	84.16
0.1	[-0.3144, 3.4560]	[0.6000, 2.5416]	94.19
0.2	[-0.1512, 3.2928]	[0.7296, 2.4120]	104.71
0.3	[0.0223, 3.1193]	[0.8511, 2.2905]	115.16
0.4	[0.2090, 2.9326]	[0.9659, 2.1757]	125.13
0.5	[0.4092, 2.7324]	[1.0750, 2.0666]	134.29
0.6	[0.6224, 2.5192]	[1.1794, 1.9621]	142.34
0.7	[0.8476, 2.2940]	[1.2803, 1.8612]	148.99
0.8	[1.0826, 2.0590]	[1.3786, 1.7630]	154.01
0.9	[1.3249, 1.8167]	[1.4751, 1.6665]	156.95
1.0	[1.5708, 1.5708]	[1.5708, 1.5708]	0

## 6 Conclusion

We have presented a new fuzzy version of Euler's method for the numerical solution of differential equations with fuzzy initial values. The method we have presented has two advantages: (1) the dependency problem which arises in the classical Euler's method is studied and handled effectively; and (2) the behaviour of the approximation solution is identical with the solution of differential equations with several crisp initial values.

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