A Study in The Theory of Geometric Functions of a Complex Variable

by

Hiba Fawzi Sabba Al-Janaby

(1343811138)

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

Institute of Engineering Mathematics
UNIVERSITI MALAYSIA PERLIS

2016

UNIVERSITI MALAYSIA PERLIS

DECLARATION OF THESIS

Author's full name : **HIBA FAWZI SABBA AL-JANABY**

Date of birth : **NOVEMBER 01, 1980**

Title : **A Study in The Theory of Geometric Functions of a Complex Variable**

Academic Session : **2016**

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ACKNOWLEDGEMENT

Foremost, thankful to Allah S.W.T, the most gracious and the most merciful. With His help and blessing to complete my thesis successfully with a lot of patience and anticipation.

I would like to express my gratitude and grat thanks to my main supervisor Dr. Muhammad Zaini Ahmad for his guidance, support and his continuous interest during my research. Also, I grateful to my co-supervisor: Assoc. Professor Dr. Rabha W. Ibrahim.

My sincere appreciation to the Dean of Institute of Engineering Mathematics (IMK), Prof. Dr. Amran Ahmed. My gratitude also goes to the Postgraduate Chairperson of Institute of Engineering Mathematics (IMK), Dr. Ahmed Kadri Junoh, and the entire staff of the Institute of Engineering Mathematics (IMK).

Not forgetting, my heartfelt thanks special to my beloved mother (Buthainah Al-Azawi), my sister (Yasamine Al-Janaby) and my niece (Jumana Hussein) for their unconditional support and encouragement to pursue my studies.

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LIST OF SYMBOLS

$\mathcal{A}$	Class of normalized analytic functions f with positive coefficients in the open unit disk of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$
$\mathcal{A}_p$	Class of normalized $p-$valent functions f with positive coefficients in the open unit disk of the form $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$
$\mathcal{A}_{p,k}$	Class of normalized $p-$valent functions f with positive coefficients in the open unit disk of the form $f(z) = z^p + \sum_{n=p+k}^{\infty} a_n z^n$
$(a)_n$	Pochhammer symbol
$\mathcal{B}(x, y)$	Beta function
$\mathcal{B}_p^{\alpha,\beta;\kappa,\mu}(x, y)$	Extended beta function
$\mathbb{C}$	Complex plane
$\mathcal{CV}$	Subclass of convex functions f in $\mathcal{A}$
$\mathcal{CV}(\alpha)$	Subclass of convex functions f of order α in $\mathcal{A}$
$\mathcal{CV}_p$	Subclass of $p-$convex functions f in $\mathcal{A}_p$
$\mathcal{CCV}$	Subclass of close-to- convex functions f in $\mathcal{A}$
$\mathcal{CCV}(\alpha)$	Subclass of close-to- convex functions f of order α in $\mathcal{A}$
$\mathcal{CCV}_p$	Subclass of $p-$close-to- convex functions f in $\mathcal{A}_p$
$\overline{co}(D)$	Closed convex hull of a set
$\mathbb{D}$	Open unit disk $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
$\overline{\mathbb{D}}$	Closed unit disk $\{z \in \mathbb{C} : \lvert z\rvert \leq 1\}$
$\mathbb{D}^*$	Open punctured unit disk $\{z \in \mathbb{C} : 0 < \lvert z\rvert < 1\}$
D	Domain
D_z^α	Fractional differential operator
D^k	Ruscheweyh differential operator
D^m	Sălăgean differential operator

$\mathrm{E(q)}$	$\{\xi : \xi \in \partial\mathbb{D} : \lim_{z\to\xi} q(z) = \infty\}$
$F(a, b; c; z)$	Gauss hypergeometric function
$F_{p;\kappa,\mu}(a, b; c; z; m)$	Extended Gauss hypergeometric function
$F(z)$	Generalized Bernardi–Libera–Livingston integral operator
$f * g$	Convolution or Hadamard product of functions f and g
$G_\alpha f(z)$	Integral of the second type
$\mathcal{H}(\mathbb{D})$	Class of analytic functions f in the open unit disk $\mathbb{D}$
$\mathcal{H}[a, n]$	$\{f \in \mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...\}$
I_z^α	Fractional integral operator
I_k	Noor integral operator
$k(z)$	Koebe function
$L(a, c)$	Carlson and Shaffer operator
$\mathbb{N}$	Set of all positive integers
$\mathcal{NS}_\mathcal{H}$	Subclass of all sense-preserving harmonic functions f in the open unit disk of the form $f(z) = z - \sum_{n=2}^{\infty} \|a_n\| z^n - \sum_{n=1}^{\infty} \|b_n\| z^n, \ \ \|b_1\| < 1$
$\mathcal{Q}$	Set of functions q that is analytic and univalent on the set $\overline{\mathbb{D}} - \mathrm{E(q)}$ and such that $q'(\xi) \neq 0$ for $\xi \in \partial\mathbb{D} - \mathrm{E(q)}$
$\mathbb{R}$	Set of all real numbers
$\Re$	Real part of a complex number
$\mathcal{S}$	Class of normalized univalent functions f with positive coefficients in the open unit disk of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$
$\mathcal{ST}$	Subclass of starlike functions f in $\mathcal{A}$
$\mathcal{ST}(\alpha)$	Subclass of starlike functions f of order α in $\mathcal{A}$
$\mathcal{ST}_p$	Subclass of $p-$starlike functions f in $\mathcal{A}_p$
$\mathcal{S}_\mathcal{H}$	Class of all sense-preserving harmonic functions f in the open unit disk of the form $f(z) = z + \sum_{n=2}^{\infty} \|a_n\| z^n + \sum_{n=1}^{\infty} \|b_n\| z^n, \ \ \|b_1\| < 1$

$\mathcal{T}$	Class of normalized analytic functions f with negative coefficients in the open unit disk of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$
$\mathcal{T}_p$	Class of normalized $p-$valent functions f with negative coefficients in the open unit disk of the form $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n$
$\mathcal{T}_{p,k}$	Class of normalized $p-$valent functions f with negative coefficients in the open unit disk of the form $f(z) = z^p - \sum_{n=p+k}^{\infty} a_n z^n$
$\prec$	Subordinate to
Σ	Class of meromorphic functions f with positive coefficients in the open punctured unit disk of the form $f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n$
Σ_p	Class of meromorphic multivalent functions f with positive coefficients in the punctured unit disk of the form $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n$
Σ^*	Class of meromorphic functions f with negative coefficients in the open punctured unit disk of the form $f(z) = z^{-1} - \sum_{n=0}^{\infty} a_n z^n$
Σ_p^*	Class of meromorphic multivalent functions f with negative coefficients in the punctured unit disk of the form $f(z) = z^{-p} - \sum_{n=p}^{\infty} a_n z^n$
$\Sigma_{\mathcal{CV}}(p,\gamma)$	Subclass of meromorphically $p-$valent convex functions f of order γ in Σ_p
$\Sigma_{\mathcal{ST}}(p,\gamma)$	Subclass of meromorphically $p-$valent starlike functions f of order γ in Σ_p
$\phi(a,c;z)$	Incomplete beta function
$\Psi_n[\Omega,q]$	Class of admissible functions

Kajian di dalam Teori Fungsi Geometri bagi Pembolehubah Kompleks

ABSTRAK

Tesis ini mengkaji beberapa jenis fungsi geometri analisis di dalam cakera unit terbuka seperti normal, meromorphic, $p-$valen, harmonik dan fungsi analitik pecahan. Lima masalah dibincangkan. Pertama, kelas fungsi analitik berperingkat pecahan dicadangkan dan digunakan untuk mentakrifkan pengoperasi pembezaan pecahan teritlak, yang mana ia sepadan dengan pengoperasi Srivastava–Owa. Dengan menggunakan konsep subordinasi dan superordinasi peringkat pertama, sempadan atas dan bawah bagi fungsi analitik pecahan yang mengandungi pengoperasi ini dibincangkan. Seterusnya, sempadan pekali untuk subkelas baru fungsi analitik multivalen ($p-$valen) yang mengandungi pengoperasi linear tertentu turut dibincangkan. Lain-lain sifat geometri bagi kelas ini juga dikaji. Satu subkelas baru bagi fungsi meromorphic $p-$valen yang ditakrifkan melalui subordinasi dan konvolusi juga diwujudkan, dan beberapa sifat geometrinya turut dikaji. Bagi sesuatu fungsi normal, perluasan fungsi hipergeometri Gauss, iaitu perluasan pengoperasi kamiran yang mengandungi pengoperasi kamiran Noor diwujudkan dan dibincangkan. Beberapa subkelas fungsi analitik yang mengandungi perluasan pengoperasi kamiran ditakrifkan dan diwujudkan. Selain itu, beberapa keputusan sandwich diperolehi. Hasil subordinasi pembezaan peringkat ketiga untuk pengoperasi linear yang berkonvolusi pengoperasi kamiran pecahan dengan fungsi beta tak lengkap yang berkaitan dengan fungsi hipergeometri pecahan, juga dikaji. Konsep dual bagi superordinasi pembezaan peringkat ketiga juga dipertimbangkan untuk mendapatkan pembezaan peringkat ketiga jenis sandwich. Keputusan diperolehi dengan menentukan kelas-kelas yang sesuai bagi fungsi yang dibenarkan untuk fungsi-fungsi pembezaan peringkat ketiga. Fasa akhir tesis ini memperkenalkan dua subkelas $\mathcal{S}_{\mathcal{H}}$, iaitu $\mathcal{L}_{\mathcal{H}}(\gamma)$ dan $\mathcal{H}(\alpha, \beta)$. Sempadan pekali, titik ekstrem, konvolusi, kombinasi cembung dan tutupan di bawah satu pengoperasi kamiran dikaji. Perhubungan di antara fungsi univalen dan fungsi hipergeometri dikaji sepenuhnya.

A Study in The Theory of Geometric Functions of a Complex Variable

ABSTRACT

This thesis deals with various types of analytic geometric functions in the open unit disk, such as normalized, meromorphic, $p-$valent, harmonic, and fractional analytic functions. Five problems are discussed. First, the class of analytic functions of fractional power is suggested and used to define a generalized fractional differential operator, which corresponds to the Srivastava–Owa operator. The upper and lower bounds for fractional analytic functions containing this operator are discussed by employing the first-order subordination and superordination. Coefficient bounds for the new subclass of multivalent ($p-$valent) analytic functions containing a certain linear operator are then presented. Other geometric properties of this class are studied. A new subclass of meromorphic $p-$valent functions defined by subordination and convolution is also established, and some of its geometric properties are studied. For a normalized function, the extended Gauss hypergeometric functions, which are generalized integral operators involving the Noor integral operator, are posed and examined. New subclasses of analytic functions containing the generalized integral operator are defined and established. In addition, some sandwich results are obtained. Third-order differential subordination outcomes for the linear operator convoluting the fractional integral operator with the incomplete beta function related to the Gauss hypergeometric function, are investigated. The dual concept of the third-order differential superordination is also considered to obtain third-order differential sandwich-type outcomes. Results are acquired by determining the appropriate classes of admissible functions for third-order differential functions. The final phase of this dissertation introduces two subclasses of $\mathcal{S}_{\mathcal{H}}$, which are denoted by $\mathcal{L}_{\mathcal{H}}(\gamma)$ and $\mathcal{H}(\alpha,\beta)$. Coefficient bounds, extreme points, convolution, convex combinations, and closure under an integral operator are investigated for harmonic univalent functions in the subclasses $\mathcal{H}(\alpha,\beta)$ and $\mathcal{L}_{\mathcal{H}}(\gamma)$. Connections between harmonic univalent and hypergeometric functions are also fully investigated.

CHAPTER 1

INTRODUCTION

1.1 Overview

Complex analysis is one of the classical branches in mathematics dating back to the 18th century. In the 20th century, important researchers on complex analysis include Euler, Gauss, Riemann, Cauchy, Weierstrass, and many more. Traditionally known as the theory of functions of a complex variable, complex analysis investigates the functions of complex numbers. It offers benefits in various branches of mathematics, including number theory and applied mathematics; as well as in physics, in the fields of hydrodynamics and thermodynamics; and in engineering fields, such as electrical engineering, mechanical, and others.

In complex analysis, a geometric function is a function whose range describes certain geometries. Geometric function theory, which studies the geometric properties of complex analytic functions, is a remarkable area in complex analysis. The cornerstone of geometric function theory is the theory of univalent functions, which is principally concerned with the analytic and univalent functions in a certain complex domain. This theory was founded in the early 20th century, when Koebe published the first important paper in this area in 1907. Alexander and Bieberbach followed suit in 1915 and 1916 respectively. In 1907, Koebe greatly contributed to the origin of univalent function theory by introducing the notion of univalent mapping or univalent functions in his monograph (Graham & Kohr, 2003).

Riemann mapping theorem is one of the most significant results in geometric function theory. Prior to Koebe, in 1850, Riemann provided this important result in geometric function theory by proving that an analytic function always exists; this function maps a simply connected domain onto another simply connected domain in a complex plane. This original version of the Riemann mapping theorem led to the birth of geometric function theory. Koebe launched the study of univalent functions in 1907, and then, in view of the Riemann mapping theorem, began the study of the properties of analytic and univalent functions on the unit open disk, rather than a general simply connected domain (Baernstein, Drasin, Duren & Marden, 1986). This field comprised the theoretical study of coefficient bounds, growth theorem, distortion theorem, differential subordination, differential superordination, and so forth. Many scholars, most notably Miller and Mocanu (2000; 2003), have performed extensive work in this field. Lately, new related topics appeared and developed interesting results and applications. The major and most interesting topic is the theory of harmonic functions, which is a natural generalization of univalent functions (Clunie & Sheil-Small, 1984).

1.2 Background of Study

The analytic function is one of the miracles of complex analysis. The fact that an analytic function in an open unit disk can be represented by a convergent power series with real or complex coefficients makes it a significant element in the study of geometric function theory. The class $\mathcal{A}$ of all analytic functions in the open unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$, and containing the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ was introduced. The subclass of functions

$\mathcal{S} \subset \mathcal{A}$, which consists of all analytic, univalent, and normalized functions in $\mathbb{D}$, also becomes the center of the study of univalent function theory. This class of functions has drawn considerable attention from various researchers around the world. The crucial property of functions in $\mathcal{A}$ is that the image domain $f(\mathbb{D})$ will describe different nice geometric properties, such as convex and starlike.

Univalent function theory is categorized under the more comprehensive area of geometric function theory. One of the main problems in the theory of univalent functions is Bieberbach's conjecture, which was proposed by Bieberbach in 1916 (Duren, 1983). This conjecture states the upper bounds for the coefficients of functions in the class $\mathcal{S}$. Bieberbach was the first to establish the bound for the second coefficient of functions in the class $\mathcal{S}$, that is, $|a_2| \leq 2$ for $f \in \mathcal{S}$. He assumed that if $f \in \mathcal{S}$, then the coefficients a_n of f will satisfy $|a_n| \leq n$, for all $n \geq 2$ (Duren, 1983). For many years, this conjecture remained a challenge to mathematicians. Finally, in 1984, Louis De Branges (1984) proved Bieberbach's conjecture (now known as de Branges theorem) by using hypergeometric functions.

The long gap between the formulation of Bieberbach's conjecture in 1916 (Duren, 1983) and its proof by De Branges (1984) motivated researchers to introduce certain classes defined by geometric conditions, such as the classes of convex functions and starlike functions. Since that time, the estimation of the coefficients $|a_n|$, for all $n \geq 2$, has been investigated for such subclasses of class $\mathcal{S}$ to provide some of the basic properties of univalent functions.

The natural generalization of univalent function is a $p-$valent (or multivalent) function, which belongs to the class $\mathcal{A}_p$, $(p \in \mathbb{N})$ and is defined in the unit disk. If f is the $p-$valent function with $p = 1$, then f is the univalent function. Aside from this, researchers have introduced subclasses of multivalent analytic functions, which are multivalently convex and multivalently starlike. The study of other classes of analytic, univalent, or meromorphic functions also started to take shape, and has remained a subject of wide interest today.

One of the most important tools mainly used in the definition of various classes of functions is the concept of subordination between analytic functions as put forward by Lindelőf in 1908 (Duren, 1983). Later, Littlewood (1925; 1944) and Rogosinski (1939; 1943) presented the term and established the basic results involving subordination. A substantial theory was developed over the years. In contrast, the study of operators plays a major role in mathematics, especially in geometric function theory. Libera (1965) introduced an integral operator and examined specific properties of starlike functions under the said operator. Sălăgean (1983) studied the class of analytic functions defined by a differential operator. These works opened new ways of studying the operators in geometric function theory. Hence, numerous studies have been conducted, all of which attempt to generalize and define different subclasses of analytic functions involving operators (Ruscheweyh, 1975; Sălăgean, 1983, Noor, 1999).

Miller and Mocanu (2000) used the principle of subordination to study various subordination theorems involving certain operators for analytic functions. In addition, Bulboaca (2002a; 2002b) and Miller and Mocanu (2003) extended the study to differential superordination as the dual problem of differential subordination. Since then, hundreds of papers regarding this topic have appeared in literature, and the applications and extensions of the theory have been developed in numerous fields, such as differential equations, partial differential equations, meromorphic functions, harmonic functions, integral operators, Banach spaces, and several complex variable functions.

Harmonic univalent mappings are closely related. These functions are widely known to have a plethora of applications in the seemingly diverse fields of engineering, physics, electronics, medicine, operations research, aerodynamics, and other branches of applied mathematical sciences. Despite offering a natural generalization to studies on analytic univalent functions, the harmonic function surprisingly needed a long period of time to capture the interest of function theorists. The turning point came with the seminal paper by Clunie and Sheil-Small (1984). In their studies, they introduced class $\mathcal{S}_{\mathcal{H}}$, which consists of normalized harmonic univalent functions defined on the open unit disk, and managed to find viable analogues of the classical coefficient bounds, growth and distortion theorems, and covering theorems for the general setting of harmonic functions. Since then, harmonic mappings have motivated function theorists to investigate other subclasses of harmonic univalent functions in addition to its geometric properties.

Utilizing hypergeometric functions in proofing Bieberbach's conjecture by De Branges (1984) has given function theorists a renewed drive to study this special function from the perspective of geometric function theory. This theory has been developed with various applications and generalizations by notable complex analysts. The hypergeometric function and its generalizations are applied in introducing various subclasses of univalent functions and obtaining several properties. Thus, the connections between analytic univalent and hypergeometric functions have been well explored, whereas few investigations have been conducted on analogous connections between hypergeometric functions and harmonic functions.

1.3 Problem Statement

The study of upper and lower bounds of fractional differential operators and linear operators in geometric function theory is very limited. Therefore, in this study, we focused on utilizing different methods to obtain these bounds. Thus far, two works have been published on the theory of third-order differential subordination and superordination, namely, Antonion and Miller (2011) and Tang, Srivastiva, Li and Ma L (2014). Third-order differential subordination and superordination are very important in the theory of geometric function. Therefore, we proposed a new study in this direction to develop the previous works. However, the connections between hypergeometric functions and harmonic functions is suggested by Ahuja and Silverman (2004). In the current study, we study the problem extensively and in depth.

1.4 Objectives

This study aims to investigate the various results in geometric function theory. The objectives are as follows:

i. To impose the upper and lower bounds for fractional analytic functions.

ii. To examine the various properties of a certain subclass of $p-$valent functions.

iii. To introduce new subclasses of analytic functions involving the Noor integral operator.

iv. To apply third-order differential subordination and superordination on a new Carlson–Shaffer operator type.

v. To investigate the connection between harmonic functions and hypergeometric functions**.**

1.5 Scope of Study

This work involves the study of a class of fractional analytic functions and certain subclasses of analytic, multivalent, multivalent meromorphic, and harmonic univalent functions by using the techniques of subordination and convolution.

i. A new class $\mathcal{A}_\alpha$ of the fractional analytic function in the open unit disk $\mathbb{D}$ is investigated as follows:

The fractional koebe function is modified by utilizing Taylor series expansion (Ibrahim, 2010) as follows:

$$h(z) = \frac{1}{\Gamma(2+\alpha)} \frac{z^{\alpha+1}}{(1-z)^{\alpha}} = \frac{1}{\Gamma(2+\alpha)} z^{1+\alpha} + \sum_{n=2}^{\infty} \mu_n z^{n+\alpha}, \tag{1.1}$$

$$\left(0 \leq \alpha < 1,\ \mu_1 := \tfrac{1}{\Gamma(2+\alpha)},\ z \in \mathbb{D}\right).$$

A class of fractional analytic functions containing $h(z)$ of the form (1.1) is denoted by $\mathcal{A}_\alpha$. This study aims to apply methods based on the first-order differential subordination and superordination of a class $\mathcal{A}_\alpha$, which involves fractional derivative operators in the sense of Srivastava–Owa operators. Several differential subordination and superordination results are obtained, and a differential sandwich-type result is also studied.

ii. A new subclass $\wp_p(\tau,\alpha,\beta)$ of class $\mathcal{T}_{p,k}$ of $p-$valent (multivalent) functions with negative coefficients is defined by utilizing a certain linear operator $D^{\gamma,\beta}_{p,m}f(z)$ as follows:

A function $f \in \mathcal{T}_{p,k}$ is said to be in the subclass $\wp_p(\tau,\alpha,\beta)$ if it satisfies the following inequality:

$$\left|\frac{(D^{\gamma,\beta}_{p,m}\,f(z))''z^{2-p}-p(p-1)}{\Big(2\tau(1-\alpha)(D^{\gamma,\beta}_{p,m}\,f(z))'z^{1-p}\Big)+\Big((D^{\gamma,\beta}_{p,m}\,f(z))''z^{2-p}-p(p-1)\Big)}\right|<\beta, \quad (1.2)$$

$$\big(0<\tau\leq 1;\ 0\leq\alpha<1;\ 0<\beta\leq 1,\ z\in\mathbb{D}\big)$$

where linear operator $D^{\gamma,\beta}_{p,m}f(z)$ (Mahzoon & Latha, 2009a) is given by

$$D^{\gamma,\beta}_{p,m}f(z)=z^p-\sum_{n=p+k}^{\infty}\Big(1+\frac{(n-p)\gamma}{(p+\beta)}\Big)^m a_n\,z^n, \quad (z\in\mathbb{D}). \quad (1.3)$$

In this study, coefficient bounds, growth and distortion theorem, radius of convexity and close-to-convexity, closure theorem, neighborhood property, partial sums, and integral means inequalities are obtained.

iii. A certain subclass $\mathcal{S}^*_g(A,B,\alpha,p,j)$ of the class Σ^*_p of meromorphically $p-$valent (multivalent) functions with negative coefficients is defined by utilizing convolution and subordination concepts and is introduced as follows.

A function $f\in\Sigma^*_p$ is said to be in the class $\mathcal{S}^*_g(A,B,\alpha,p,j)$ if it satisfies the following subordination condition:

$$\frac{z\,((f*g)(z))^{(j+1)}}{((f*g)(z))^{(j)}}\prec-\left[\frac{(p+j)+[B(p+j)-(p+j-\alpha)(A-B)]\,z}{1+Bz}\right], \quad (1.4)$$

$$\big(-1\leq A<B\leq 1;\ 0<B\leq 1;\ 0\leq\alpha<p+j;\ j\leq p\leq n;\ p\in\mathbb{N};\ j\in 2\mathbb{N}\cup\{0\}\big),$$

for the function