

# Derivation of the Matrix Equation for a Translational Mechanical System with Three Degrees of Freedom

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## ABSTRACT

*Several researchers have carried out derivation of equation of motion of mechanical systems with more than one degrees of freedom of movement using different approaches among which is the work of Sivak and Darina [11] who derived the equation of motion of a translational mechanical system with two degrees of freedom using Newton's second law. This paper, therefore, provides an extension of the work of Sivak and Darina [11] to model a three degree of freedom translational mechanical system. The free-body diagrams of the individual masses are developed and then Newton's second law applied. Finally, the three equations derived are presented in matrix form in order to solve the system vibration problems.*

**Keywords:** Degree-of-freedom, Differential equation, Matrix representation, Mechanical system.

## 1 INTRODUCTION

In dynamical systems, the existing tools for analysis and synthesis are based on certain types of mathematical descriptions of the system in question (Christiansen) [1]. For this purpose, according to [2], an engineer requires a mathematical model of the plant in order to design a system which will accomplish the desired objectives. According to [3], differential equations can describe the dynamic behavior of any system. By using physical laws governing a specific system, one may be able to derive such differential equations, among them are Newton's laws for mechanical systems and Kirchoff's laws for electrical ones.

It is not uncommon for a modern complex system to have many inputs and many outputs (MIMO), which may be interrelated in a complex manner. Such systems can be analyzed using a system of simultaneous equations for output and input based on the principle of superposition. In this type of description, a system is described from its input and output terminals. To solve vibration problems in mechanical systems using this approach, a matrix can be used in the presentation of the system of equations. Tejal [4], pointed out that the dynamic behavior of any system can be determined by using differential equations for the output of the system.

Adding that, when solved explicitly, an understanding of the system can be gained from the equations. According to [5] and [6], modeling of systems with multiple inputs and output has been a topic of much research activity for the last few decades. Systems with two or more degrees of freedom and significantly different levels of energy dissipation are common in dynamical systems [7]. Geyer et al. [8] and Whittington and Thelen [9] pointed out that modeling of MIMO mechanical systems developed from simple mass spring models to relatively complex models with relatively high degrees of freedom. Kuo [10] revealed that the inclusion of multi-degree of freedom in a mechanical system model provides more precise results in most cases, but also much effort is needed to figure out how the system works internally and what is going on with internal components.

In the above studies, time domain differential equation and matrix representation method is worthy to be noted for high efficiency regarding the vibration and analysis of mechanical systems. The application of this approach is simple and unique for cases of multi-degree of freedom systems. Sivak and Darina [11], worked on a two-degree-of-freedom translational mechanical system (single-input, multiple-output (SIMO)). This work successfully carried out the formulation of the equation of motion of the system limited to two degrees of freedom. However, matrix representation of the system of equations was not carried out. This work therefore, will deal with a three-degree-of-freedom translational mechanical system and in addition to the work of Sivak and Darina [11], the work will present the model in matrix form.

Figure 1 depicts a translational mechanical system with three degrees of freedom of movement that is the subject of this paper. This system is an extension of the work of Sivak and Darina [11] which dealt with a translational mechanical system with two degrees of freedom. The inclusion of one more degree of freedom will make the results of the analysis more precise. The system has bodies of masses  $M_1$ ,  $M_2$  and  $M_3$  with stiffnesses  $K_1$  and  $K_2$ , and dampers  $B_1$  and  $B_2$ . The masses are not connected to any rigid frame and the system moves linearly in the direction of springs and dampers. The springs weights are not taken into account. For the excitation of the model, a step input force is applied, with the respective masses performing a linear forced oscillation.

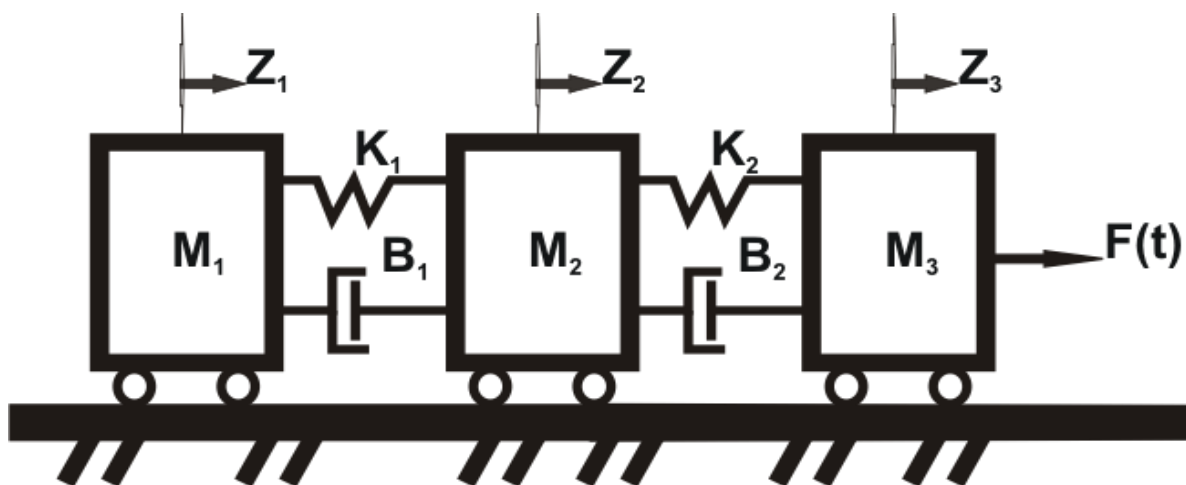


Figure 1: Original damped three degrees of freedom system model

The aim of the study is to describe the mathematical representation of the translational mechanical system in Figure 1 and the objective is to derive the system's equation of motion using Newton's second law and present the equations in matrix form.

## 2 DERIVING THE SYSTEM'S EQUATIONS OF MOTION

According to [12], the equation of motion of a system with n-degrees of freedom is paired with normal second-order ordinary differential equations. Newton's second law will be utilized to derive the equations of the system in question, in time domain. In order to accomplish this, the free-body diagrams of the masses that make up the system will be constructed as follows:

### 2.1 General Free-body Diagrams for n-degrees of Freedom

It can be seen from Figure 1 that the coordinates,  $Z_1$ ,  $Z_2$  and  $Z_3$ , of  $M_1$ ,  $M_2$  and  $M_3$  respectively, as they were measured from their respective equilibrium positions describe the position of the masses.

Furthermore, Figure 2 shows the general free-body diagrams for any number of masses  $i = 1, 2, \dots$  in the system along with their displacement, velocity and acceleration in positive direction.

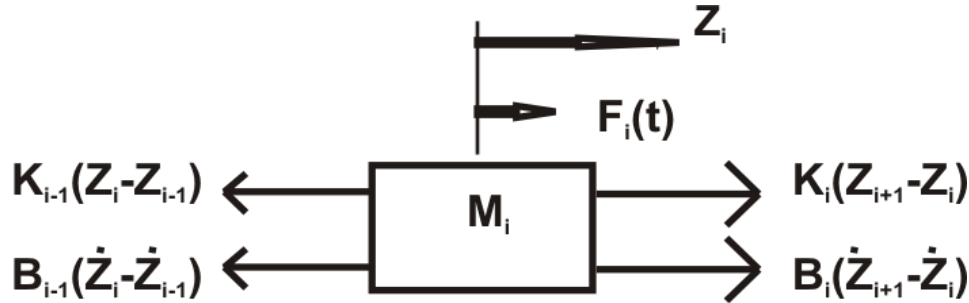


Figure 2: General free-body diagrams of the system

Thus, the application of Newton's second law of motion to mass  $M_i$  gives:

$$M_i \ddot{Z}_i = -K_{i-1}(Z_i - Z_{i-1}) + K_i(Z_{i+1} - Z_i) - B_{i-1}(\dot{Z}_i - \dot{Z}_{i-1}) + K_i(\dot{Z}_{i+1} - \dot{Z}_i) + F_i \quad (1)$$

From (1), the equations of motion for the masses can be derived by setting  $i = 1$  together with  $Z_0 = 0$ ,  $K_0 = 0$ ,  $B_0 = 0$  and  $i = 3$  together with  $Z_4 = 0$ ,  $K_3 = 0$ ,  $B_3 = 0$ , respectively as shown in Figure 3.

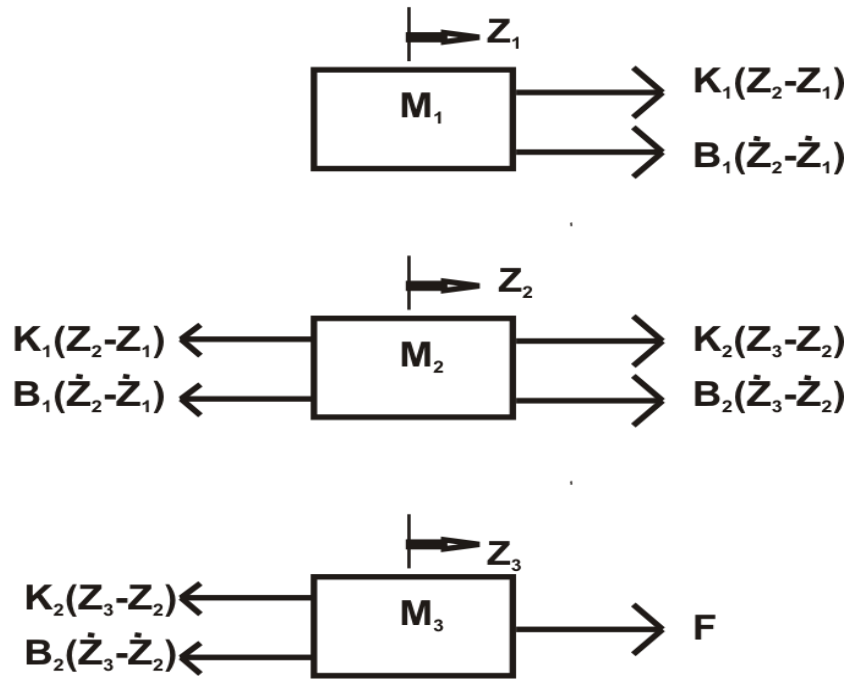


Figure 3: Three free-body diagrams for M1, M2, M3

By Newton's second law, the equations are written as:

$$M_1 \ddot{Z}_1 = -K_0(Z_1 - Z_0) + K_1(Z_2 - Z_1) - B_0(\dot{Z}_1 - \dot{Z}_0) + B_1(\dot{Z}_2 - \dot{Z}_1) + F_1 \quad (2)$$

$$M_2 \ddot{Z}_2 = -K_1(Z_2 - Z_1) + K_2(Z_3 - Z_2) - B_1(\dot{Z}_2 - \dot{Z}_1) + B_2(\dot{Z}_3 - \dot{Z}_2) + F_2 \quad (3)$$

$$M_3 \ddot{Z}_3 = -K_2(Z_3 - Z_2) + K_3(Z_4 - Z_3) - B_2(\dot{Z}_3 - \dot{Z}_2) + B_3(\dot{Z}_4 - \dot{Z}_3) + F_3 \quad (4)$$

But the input force function in the system is only applied on  $M_3$ . Hence,

$$F_1 = F_2 = 0 \quad (5)$$

Therefore, equations (2), (3) and (4) resolve to:

$$M_1 \ddot{Z}_1 = -K_0(Z_1 - Z_0) + K_1(Z_2 - Z_1) - B_0(\dot{Z}_1 - \dot{Z}_0) + B_1(\dot{Z}_2 - \dot{Z}_1) \quad (6)$$

$$M_2 \ddot{Z}_2 = -K_1(Z_2 - Z_1) + K_2(Z_3 - Z_2) - B_1(\dot{Z}_2 - \dot{Z}_1) + B_2(\dot{Z}_3 - \dot{Z}_2) \quad (7)$$

$$M_3 \ddot{Z}_3 = -K_2(Z_3 - Z_2) + K_3(Z_4 - Z_3) - B_2(\dot{Z}_3 - \dot{Z}_2) + B_3(\dot{Z}_4 - \dot{Z}_3) + F_3 \quad (8)$$

Simplifying the above equations yields:

$$M_1 \ddot{Z}_1 = K_1(Z_2 - Z_1) + B_1(\dot{Z}_2 - \dot{Z}_1) \quad (9)$$

$$M_2 \ddot{Z}_2 = -K_1(Z_2 - Z_1) + K_2(Z_3 - Z_2) - B_1(\dot{Z}_2 - \dot{Z}_1) + B_2(\dot{Z}_3 - \dot{Z}_2) \quad (10)$$

$$M_3 \ddot{Z}_3 = -K_2(Z_3 - Z_2) - B_2(\dot{Z}_3 - \dot{Z}_2) + F_3 \quad (11)$$

Simplifying again, the equations of motion of the system are obtained as follows:

$$M_1 \ddot{Z}_1 + B_1 \dot{Z}_1 - B_1 \dot{Z}_2 + K_1 Z_1 - K_1 Z_2 = 0 \quad (12)$$

$$M_2 \ddot{Z}_2 - B_1 \dot{Z}_1 + (B_1 + B_2) \dot{Z}_2 - B_2 \dot{Z}_3 - K_1 Z_1 + (K_1 + K_2) Z_2 - K_2 Z_3 = 0 \quad (13)$$

$$M_3 \ddot{Z}_3 - B_2 \dot{Z}_2 + B_2 \dot{Z}_3 - K_2 Z_2 + K_2 Z_3 = F_3 \quad (14)$$

Hence, Equations (12), (13) and (14) form the system of equations that will accurately or at least fairly well represent the dynamics of the system. This is a set of three equations that are second order ordinary differential equations (ODEs) that require knowledge about the initial states of the position and velocity for all three degrees of freedom in order to solve for the transient response of the system.

### 3 MATRIX REPRESENTATION OF THE SYSTEM

Equations (12), (13) and (14) above are systemically presented in the order which can be represented in matrix form as follows [13]:

$$M\ddot{Z} + B\dot{Z} + KZ = F \quad (15)$$

Where **M**, **B**, and **K** are the matrices for mass, damping, and stiffness, respectively, expressed as follows:

$$M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} \quad (16)$$

$$\mathbf{B} = \begin{pmatrix} B_1 & -B_1 & 0 \\ -B_1 & (B_1 + B_2) & -B_2 \\ 0 & -B_2 & B_2 \end{pmatrix} \quad (17)$$

$$\mathbf{K} = \begin{pmatrix} K_1 & -K_1 & 0 \\ -K_1 & (K_1 + K_2) & -K_2 \\ 0 & -K_2 & K_2 \end{pmatrix} \quad (18)$$

While the displacement, velocity, acceleration and force vectors  $\mathbf{Z}$ ,  $\dot{\mathbf{Z}}$ ,  $\ddot{\mathbf{Z}}$  and  $\mathbf{F}$  respectively, are:

$$\mathbf{Z} = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} \quad (19)$$

$$\dot{\mathbf{Z}} = \begin{pmatrix} \dot{Z}_1 \\ \dot{Z}_2 \\ \dot{Z}_3 \end{pmatrix} \quad (20)$$

$$\ddot{\mathbf{Z}} = \begin{pmatrix} \ddot{Z}_1 \\ \ddot{Z}_2 \\ \ddot{Z}_3 \end{pmatrix} \quad (21)$$

$$\mathbf{F} = \begin{pmatrix} 0 \\ 0 \\ F_3 \end{pmatrix} \quad (22)$$

Hence, the matrix equation of motion for the system is written as follows:

$$\begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} \begin{pmatrix} \ddot{Z}_1 \\ \ddot{Z}_2 \\ \ddot{Z}_3 \end{pmatrix} + \begin{pmatrix} B_1 & -B_1 & 0 \\ -B_1 & (B_1 + B_2) & -B_2 \\ 0 & -B_2 & B_2 \end{pmatrix} \begin{pmatrix} \dot{Z}_1 \\ \dot{Z}_2 \\ \dot{Z}_3 \end{pmatrix} + \begin{pmatrix} K_1 & -K_1 & 0 \\ -K_1 & (K_1 + K_2) & -K_2 \\ 0 & -K_2 & K_2 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = (1) \begin{pmatrix} 0 \\ 0 \\ F_3 \end{pmatrix} \quad (23)$$

As a result of the  $\mathbf{M}$ ,  $\mathbf{B}$ , and  $\mathbf{K}$  matrices in the model, the model has the capability of handling the single input, multiple output (SIMO) situation. Also, computers can easily manipulate the system because it is compactly represented by matrices [14]. This approach maximizes computational accuracy, efficiency, and programming convenience [15].

#### 4 CONCLUSION

In this paper, a translational mechanical system with three degrees of freedom was model using time-domain differential equation approach. The system was first presented in equivalent free body diagrams. Newton's second law of motion was then applied to derive the equations of motion and subsequently presented the equations in matrix form. The model systematically handled the translational mechanical system as a single input multiple output (SIMO) system. Thus, the model will accurately or fairly-well represent the dynamics of the system.

The model can be used to identify the system's most important properties (eigenvalues and eigenvectors) for various masses, stiffnesses, and dampers which will help in solving problems commonly encountered by mechanical engineers (for example, resonance problem). With the knowledge of these properties, an engineer will be able to adjust the dynamics of the system in order to achieve stability and balance in the system structure.

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