

A Family of Third Order Iterative Methods for Solving Nonlinear Equations Free of Second Derivative

Muhammad Shakur Ndayawo^{1*} and Babangida Sani²

¹Department of Mathematics and Statistics, Kaduna Polytechnic, Kaduna Nigeria

²Department of Mathematics, Ahmadu Bello University Zaria, Nigeria

*Corresponding author: nmshakurn@kadunapolytechnic.edu.ng

Received: 16 May 2022; Accepted: 28 Jun 2022; Available online (In press): 02 August 2022

ABSTRACT

In this paper, we proposed and analysed a family of iterative methods for solving nonlinear problems. The methods have been developed by applying Adomian decomposition method to Taylor's series expansion. Using one-way ANOVA, the methods are compared with other existing methods in terms of number of iterations and solution to convergence between the individual methods used. Numerical examples are used in the comparison to justify the efficiency of the new iterative methods.

Keywords: ANOVA, Iterative methods, Nonlinear equations, Order of Convergence

1 INTRODUCTION

Finding the roots of higher order algebraic polynomials, exponential or transcendental equations, has always been an interesting problem in physics, solid mechanics, astrophysics, mathematics, engineering and other disciplines. Appropriate and convenient mathematical models have been developed and used to find the approximate solutions of these nonlinear problems. This is because analytic solutions for such problems are not always readily available. In practice, one can give approximate solutions that are close to the analytic solutions. Newton's method (popularly known as Newton-Raphson method) and its variants are popular methods used for solving such problems. The Newton method approximates the root of an equation in one variable, using the value of the function and its derivatives, in an iterative way. The Newton's method approximately doubles the number of significant digits at every iteration carried out. Recently, different authors have developed new and better iterative methods with faster rates of convergence for solving nonlinear equations. Some of these methods were developed by applying Adomian decomposition method [1], which decomposes any given nonlinear equation into solutions given in a series form. The series is obtained in a recursive manner from polynomials known as the Adomian polynomials. [2] developed a decomposition technique used to obtain solutions of nonlinear functional equations. This decomposition method was used by [3] to propose a three step iterative method with third order rate of convergence for solving nonlinear equations. [4] presented a three-step iterative method with remarkable improvement over that method developed by [3]. The homotopy perturbation method (HPM) was developed by [5], for solving nonlinear systems. [6], combined the HPM with homotopy

analysis method to develop and analyse a class of iterative methods for solving nonlinear equations. Convergence of the method is of order four and several numerical examples were given by the authors to illustrate the efficiency and performance of these methods. [7] presented an iterative method based on both Adomian decomposition method and considering the Taylor's series expansion. The particular expressions of the method for a function $f(x)$ is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2 \frac{f(x_{n+1}^*)}{f'(x_n)} + \frac{f(x_{n+1}^*)f'(x_{n+1}^*)}{[f'(x_n)]^2}$$

where x_{n+1} is the new iterate, x_n is the previous one and $x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)}$

[8] developed a new iterative method, based on Adomian decomposition method and considering the Taylor's series expansion and rewriting the nonlinear equation as a coupled system of equations. The method which is a two-step iterative method can also be considered as predictor-corrector type method. Several numerical examples were given to illustrate the efficiency and performance of the method. The expression of the method is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - (y_n - x_n)^2 \frac{f''(x_n)}{f'(x_n)} - 2 \frac{f(y_n)}{f'(x_n)} + (y_n - x_n)^2 \frac{f'(y_n)}{2f'^2(x_n)} f''(x_n) + \frac{f(y_n)f'(y_n)}{f'^2(x_n)} \\ + (y_n - x_n) \frac{f(y_n)f''(x_n)}{f'^2(x_n)} + (y_n - x_n)^3 \frac{f''^2(x_n)}{2f'^2(x_n)}$$

[9] presented an iterative method whose convergence rate was proved to be of order three. The iterative scheme is as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x)}{2[f'(x_n)]^3 - 2f(x_n)f'(x_n)f''(x_n)}$$

[10] developed two third-order iterative methods based on Adomian decomposition method and Taylor's series expansion with the assumption that $\frac{f''(x)}{f'(x)} \approx 1$. One of the methods performed better than the other one and the essential expression used in that method is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2}{2[f'(x_n)]^2} - \frac{[f(x_n)]^3}{2[f'(x_n)]^3}$$

In this paper, we have developed a family of new iterative methods for solving nonlinear equations without second derivative. Having second or higher order derivative in an iterative method is a drawback, because evaluating second derivative at each step is cumbersome. This is particularly true

for those methods that are derived using higher order of Taylor's series. We have analysed three members of the family where two of them are of third-order rate of convergence and the third one is of fourth order. The methods are derived by considering the Taylor's series expansion around x of higher order, then applying Adomian decomposition method. The performance of the new iterative methods are analysed in terms of number of iterations in comparison with other existing schemes. The schemes used in the comparison include Newton-Raphson method (NR), [7] (CM), [8] (NM), [9] (BM) and the best of the two [10] (NSM). The rates of convergence of the proposed methods in many cases, in the test examples, are faster or equal to some of the existing schemes. Note, however that a limitation of the method and other methods above, is that one might get into trouble if one does not know exactly where the root is located. This is because in that case, it will be difficult to know x_0 .

2 MATERIAL AND METHOD

2.1 The Adomian Decomposition method

[1], developed the Adomian decomposition method in order to solve equations that can be written in a canonical form. The method has been applied to solve problems in physics (such as oscillating systems, Navier Stokes equations e.t.c) [11], astrophysics, solid mechanics, mathematics, engineering and other related fields. This method does not require any assumption or linearization to solve any given problem. The idea is as follows:

Consider the equation:

$$Fu = g(t, x),$$

where F is a differential operator involving linear and nonlinear terms. Rewrite the equation in operator form as

$$Lu + Ru + Nu = g \tag{1}$$

where L is the highest order derivative which is easily invertible, R is the remainder of the linear differential portion, and N is a nonlinear operator. Solving (1) and since L is invertible we get

$$u = L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu$$

Since F is taken to be a differential operator and L is linear, L^{-1} would represent integration with the given initial or boundary conditions, see[1].

The solutions of (1) consist of approximate solutions as an infinite series

$$u(t) = \sum_{n=0}^{\infty} u_n(t)$$

Decomposing the nonlinear term into a series of Adomian polynomials, as

$$Nu = \sum_{n=0}^{\infty} A_n$$

where A_n 's, are called the Adomian polynomials, depending on u_0, u_1, \dots, u_n , [12].

To determine the Adomian polynomials, a grouping parameter, λ is introduced. It should be noted that λ is not a “smallness parameter”. The parameter is used in determining the polynomials,

$$u(t) = \sum_{n=0}^{\infty} \lambda^n u_n \tag{2}$$

and

$$Nu(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n \tag{3}$$

which gives rise to the Adomian polynomials.

Then

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, n = 0, 1, 2, \dots$$

The first few Adomian polynomials are given as follows

$$\left. \begin{aligned} A_0 &= N(x_0), \\ A_1 &= x_1 N'(x_0), \\ A_2 &= x_2 N'(x_0) + \frac{1}{2} x_1^2 N''(x_0), \\ A_3 &= x_3 N'(x_0) + x_1 x_2 + \frac{1}{3!} x_1^3 N''(x_0), \\ &\vdots \end{aligned} \right\} \tag{3A}$$

2.1.1 Development of the new Iterative methods

Consider the nonlinear equation,

$$f(x) = 0 \tag{4}$$

If α is a root of (4), and γ is the initial guess sufficiently close to α , then (4) can be rewritten using Taylor’s series, see [13], so that

$$f(\gamma) + f'(\gamma)(x - \gamma) + f''(\gamma) \frac{(x - \gamma)^2}{2} + g(x) = 0 \quad (5)$$

where $g(x)$ represents the truncated part of higher orders from the third order. Note however that γ is just a notation and not a gamma function.

Rearranging (5) we get the following equation,

$$g(x) = f(x) - f(\gamma) - f'(\gamma)(x - \gamma) - f''(\gamma) \frac{(x - \gamma)^2}{2}, \text{ for } f(x) = 0 \quad (6)$$

and from (5) we obtain

$$\begin{aligned} f'(\gamma)(x - \gamma) &= -f(\gamma) - f''(\gamma) \frac{(x - \gamma)^2}{2} - g(x) \\ \Rightarrow x - \gamma &= -\frac{f(\gamma)}{f'(\gamma)} - \frac{f''(\gamma)}{f'(\gamma)} \frac{(x - \gamma)^2}{2} - \frac{g(x)}{f'(\gamma)} \\ \Rightarrow x &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - \mathbb{Z} \frac{(x - \gamma)^2}{2} - \frac{g(x)}{f'(\gamma)}, \end{aligned} \quad (7)$$

{on the assumption that $\frac{f''(\gamma)}{f'(\gamma)} \approx \mathbb{Z}$ where \mathbb{Z} is an integer}

$$\therefore x = c + N(x) \quad (8)$$

where $c = \gamma - \frac{f(\gamma)}{f'(\gamma)}$

and

where $N(x) = -\mathbb{Z} \frac{(x - \gamma)^2}{2} - \frac{g(x)}{f'(\gamma)}$ is a nonlinear function. (9)

Comparing equation (9) with the Adomian decomposition series solution in equations (2) and (3),

we obtain $x = \sum_{n=0}^{\infty} x_n$, where the nonlinear function is given as $N(x) = \sum_{n=0}^{\infty} A_n$

$$\Rightarrow x = \sum_{n=0}^{\infty} x_n = c + \sum_{n=0}^{\infty} A_n \text{ from which } x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}.$$

If $x = x_0 = \gamma$ for initial guess, we get

$$g(x) = f(x) - f'(\gamma)(x - \gamma) - f(\gamma) - f''(\gamma) \frac{(x - \gamma)^2}{2} \text{ from equation (6).}$$

$$\Rightarrow g(x) = f(x)$$

$$\therefore g(x_0) = f(x_0)$$

Then

$$N(x_0) = -\mathbb{Z} \frac{(x_0 - \gamma)^2}{2} - \frac{g(x_0)}{f'(\gamma)} = -\mathbb{Z} \frac{(x_0 - \gamma)^2}{2} - \frac{f(x_0)}{f'(\gamma)} \text{ at initial guess}$$

$$\Rightarrow N(x_0) = -\mathbb{Z} \frac{(x_0 - x_0)^2}{2} - \frac{f(x_0)}{f'(\gamma)} = x_1 = -\frac{f(x_0)}{f'(\gamma)} = A_0 \text{ see equation (3A).}$$

from (6) and (9) we have

$$N(x) = -\mathbb{Z} \frac{(x - \gamma)^2}{2} - \frac{f(x)}{f'(\gamma)} + \frac{f(\gamma)}{f'(\gamma)} + \frac{f'(\gamma)}{f'(\gamma)}(x - \gamma) + \mathbb{Z} \frac{(x - \gamma)^2}{2} \tag{10}$$

In considering the Taylor's series, [7] considered only the first two terms, while the remaining terms were considered as $g(x)$. However in our case, we considered the first 3 terms and made the remaining terms as $g(x)$, see equation (5). In equation (10) above the first and last terms cancel and that will take us back to Chun's method. To avoid this, however we approximate equation (10), so that the resulting expression will lead to a new iterative scheme.

$$\text{i.e. } N(x) \approx -\mathbb{Z} \frac{(x - \gamma)^2}{2} - \frac{f(x)}{f'(\gamma)} + \frac{f(\gamma)}{f'(\gamma)} + \frac{f'(\gamma)}{f'(\gamma)}(x - \gamma)$$

Differentiating with respect to x , approximate to

$$N'(x) \approx -\mathbb{Z}(x - \gamma) - \frac{f'(x)}{f'(\gamma)} + 1 \approx -\mathbb{Z} \left(\gamma - \frac{f(\gamma)}{f'(\gamma)} - \gamma \right) - \frac{f'(x)}{f'(\gamma)} + 1, \text{ since } x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)}$$

$$\therefore N'(x_0) \approx 1 + \mathbb{Z} \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)}$$

$$A_1 = x_1 N'(x_0) \approx \frac{-f(x_0)}{f'(\gamma)} \left\{ 1 + \mathbb{Z} \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)} \right\} = -\frac{f(x_0)}{f'(\gamma)} - \mathbb{Z} \frac{f(x_0)f(\gamma)}{f'^2(\gamma)} + \frac{f(x_0)f'(x_0)}{f'^2(\gamma)} = x_2$$

$$\text{Since } x_1 = \frac{-f(x_0)}{f'(\gamma)}$$

Now x is approximated by $X_m = x_0 + x_1 + \dots + x_m = x_0 + A_0 + A_1 + \dots + A_{m-1}$

where $\lim_{m \rightarrow \infty} X_m = x$.

For $m=0$

$$x = X_0 = x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For $m=1$

$$x = X_1 = x_0 + x_1 = c + A_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)}$$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_{n+1}^*)}{f'(x_n)}$ where $x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)}$ this is because $x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}$ and so

the new iterate for x_0 becomes $x_0 = x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)}$

For $m=2$

$$x = X_2 = x_0 + x_1 + x_2 = c + A_0 + A_1 = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)} - \mathbb{Z} \frac{f(x_0)f(\gamma)}{f'^2(\gamma)} + \frac{f(x_0)f'(x_0)}{f'^2(\gamma)}$$

$$= \gamma - \frac{f(\gamma)}{f'(\gamma)} - 2 \frac{f(x_0)}{f'(\gamma)} - \mathbb{Z} \frac{f(x_0)f(\gamma)}{f'^2(\gamma)} + \frac{f(x_0)f'(x_0)}{f'^2(\gamma)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2 \frac{f(x_{n+1}^*)}{f'(x_n)} - \mathbb{Z} \frac{f(x_{n+1}^*)f(x_n)}{f'^2(x_n)} + \frac{f(x_{n+1}^*)f'(x_{n+1}^*)}{f'^2(x_n)} \quad (11)$$

For any value of \mathbb{Z} we can obtain a new iterative method for solving nonlinear equations. For instance, for $\mathbb{Z} = -1, 0$ and 1 , we obtain the following schemes.

(i) for $\mathbb{Z} = -1$ we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2 \frac{f(x_{n+1}^*)}{f'(x_n)} + \frac{f(x_{n+1}^*)f(x_n)}{f'^2(x_n)} + \frac{f(x_{n+1}^*)f'(x_{n+1}^*)}{f'^2(x_n)} \quad (\text{Algorithm 2.1 say})$$

(ii) for $\mathbb{Z} = 0$ we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2 \frac{f(x_{n+1}^*)}{f'(x_n)} + \frac{f(x_{n+1}^*)f'(x_{n+1}^*)}{f'^2(x_n)} \quad (\text{Algorithm 2.2 say})$$

and (iii) for $\mathbb{Z} = 1$ we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2 \frac{f(x_{n+1}^*)}{f'(x_n)} - \frac{f(x_{n+1}^*)f(x_n)}{f'^2(x_n)} + \frac{f(x_{n+1}^*)f'(x_{n+1}^*)}{f'^2(x_n)} \quad (\text{Algorithm 2.3 say})$$

Note however, that Algorithm 2.2 is the same as [7]. This means the new iterative method is a generalisation of [7], i.e. when $\mathbb{Z} = 0$, we get [7].

3 CONVERGENCE ANALYSIS

Theorem 3.1; Let $\alpha \in I$ be a simple root of a sufficiently differentiable function $f : I \rightarrow R$ where I is an open interval. Then the New method (11) has a third order rate of convergence when $\mathbb{Z} = -1$ and 1 and has a fourth order rate of convergence when $\mathbb{Z} = 0$. The error satisfies the following error equations:

$$e_{n+1} = (\mathbb{Z}c_2 - \mathbb{Z}^2)e_n^3 + 0(e_n^4) \quad \text{for } \mathbb{Z} = -1 \text{ and } 1 \text{ and } e_{n+1} = (2\mathbb{Z}^3 - 4\mathbb{Z}^2c_2 + 3\mathbb{Z}c_3 - c_2c_3)e_n^4 + 0(e_n^5) \text{ for } \mathbb{Z} = 0$$

Proof

Let α be a simple root of $f(x)$ i.e. $f(\alpha) = 0, f'(\alpha) \neq 0$. Assume e_n to be the error at the n th iteration so that $e_n = x_n - \alpha$, By Taylors series expansion

$$f(x_n) = f(\alpha + e_n)$$

$$\begin{aligned}
 &= f(\alpha) + f'(\alpha)e_n + \frac{f''(\alpha)}{2!}e_n^2 + \frac{f'''(\alpha)}{3!}e_n^3 + \frac{f^{(4)}(\alpha)}{4!}e_n^4 + 0(e_n^5) \\
 &= f'(\alpha) \left[e_n + \frac{f''(\alpha)}{2f'(\alpha)}e_n^2 + \frac{f'''(\alpha)}{3!f'(\alpha)}e_n^3 + \frac{f^{(4)}(\alpha)}{4!f'(\alpha)}e_n^4 + 0(e_n^5) \right], \text{ since } f(\alpha) = 0, \\
 \therefore f(x_n) &= f'(\alpha) \left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + 0(e_n^5) \right], \\
 \Rightarrow f'(x_n) &= f'(\alpha) \left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 0(e_n^4) \right],
 \end{aligned} \tag{12}$$

where $c_j = \frac{1}{j!} \frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 2, 3, \dots$

$$\text{i.e. } c_2 = \frac{f''(\alpha)}{2f'(\alpha)} \Rightarrow 2c_2 = \mathbb{Z} \text{ since } \frac{f''(\alpha)}{f'(\alpha)} \approx \mathbb{Z} \text{ and } e_n = x_n - \alpha \tag{13}$$

$$\therefore f'(x_n) = f'(\alpha) \left[1 + \mathbb{Z}e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_4e_n^4 + 0(e_n^4) \right] \tag{14}$$

$$\text{and } f^{(2)}(x_n) = f^{(2)}(\alpha) \left[1 + 2\mathbb{Z}e_n + (6c_3 + \mathbb{Z}^2)e_n^2 + (6\mathbb{Z}c_3 + 8c_4)e_n^3 + (0)e_n^4 \right] \tag{15}$$

From equations (12) and (14) we get

$$\begin{aligned}
 \frac{f(x_n)}{f'(x_n)} &= \left\{ e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + 0(e_n^5) \right\} \left\{ 1 + \mathbb{Z}e_n + 3c_3e_n^2 + 4c_4e_n^3 + 0(e_n^4) \right\}^{-1} \\
 &= \left\{ e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + 0(e_n^5) \right\} \left\{ 1 - [\mathbb{Z}e_n + 3c_3e_n^2 + 4c_4e_n^3 + 0(e_n^4)] + [\mathbb{Z}e_n + 3c_3e_n^2 + 4c_4e_n^3 + 0(e_n^4)]^2 - \dots \right\} \\
 &= e_n + (c_2 - \mathbb{Z})e_n^2 + (-\mathbb{Z}c_2 - 2c_3 + \mathbb{Z}^2)e_n^3 + 0(e_n^4) \text{ after simplification.}
 \end{aligned}$$

However,

$$x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1}^* = \alpha + e_n - \left[e_n + (c_2 - \mathbb{Z})e_n^2 + (-\mathbb{Z}c_2 - 2c_3 + \mathbb{Z}^2)e_n^3 + 0(e_n^4) \right] \tag{16}$$

and so

$$x_{n+1}^* = \alpha + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 - (c_2 - \mathbb{Z})e_n^2 + 0(e_n^4)$$

Thus the error for x_{n+1}^* is

$$(\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 - (c_2 - \mathbb{Z})e_n^2 + 0(e_n^4) \tag{17}$$

From equation (12), we can obtain the value of $f(x_{n+1}^*)$ by substituting e_n in equation (17), and we get:

$$\begin{aligned} f(x_{n+1}^*) &= f'(\alpha) \left\{ -(c_2 - \mathbb{Z})e_n^2 + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 + c_2 \left[-(c_2 - \mathbb{Z})e_n^2 + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 \right]^2 + 0(e_n^4) \right\} \\ &= f'(\alpha) \left\{ -(c_2 - \mathbb{Z})e_n^2 + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 + 0(e_n^4) \right\} \end{aligned} \tag{18}$$

Similarly, using equation (14), we can obtain the value of $f'(x_{n+1}^*)$, after substituting e_n in equation (17).

This gives:

$$\begin{aligned} f'(x_{n+1}^*) &= f'(\alpha) \left\{ 1 - \mathbb{Z} \left[(c_2 - \mathbb{Z})e_n^2 + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 + 0(e_n^4) \right] + 3c_3 \left[\begin{matrix} -(c_2 - \mathbb{Z})e_n^2 + \\ (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 + 0(e_n^4) \end{matrix} \right]^2 + \dots \right\} \\ &= f'(\alpha) \left\{ 1 - \left[(\mathbb{Z}c_2 - \mathbb{Z}^2)e_n^2 + (\mathbb{Z}^2c_2 + 2\mathbb{Z}c_3 - \mathbb{Z}^3)e_n^3 \right] + 0(e_n^4) \right\} \end{aligned} \tag{19}$$

Using equations (14) and (18) we get

$$\begin{aligned} \frac{f(x_{n+1}^*)}{f'(x_n)} &= \frac{(\mathbb{Z} - c_2)e_n^2 + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 + (\mathbb{Z}^3 - 2\mathbb{Z}c_2^2 + c_2^3 - 5\mathbb{Z}c_3 + 3c_2c_3 + 3c_4)e_n^4 + 0(e_n^5)}{1 + \mathbb{Z}e_n + 3c_3e_n^2 + 4c_4e_n^3 + 0(e_n^4)} \\ &= \left(-(c_2 - \mathbb{Z}) + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 + (\mathbb{Z}^3 - 2\mathbb{Z}c_2^2 + c_2^3 - 5\mathbb{Z}c_3 + 3c_2c_3 + 3c_4)e_n^4 + 0(e_n^5) \right) \\ &\quad \left(1 + \mathbb{Z}e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 0(e_n^4) \right)^{-1} \\ &= (\mathbb{Z} - c_2)e_n^2 + (2\mathbb{Z}c_2 + 2c_3 - 2\mathbb{Z}^2)e_n^3 + (3\mathbb{Z}^3 - 2\mathbb{Z}^2c_2 - 2\mathbb{Z}c_2^2 + c_2^3 - 10\mathbb{Z}c_3 + 6c_2c_3 + 3c_4)e_n^4 + 0(e_n^5) \end{aligned}$$

after simplification.

Since $2c_2 = \mathbb{Z}$ we have the last equation becoming

$$\begin{aligned} &= (\mathbb{Z} - c_2)e_n^2 + (2c_3 - \mathbb{Z}^2)e_n^3 + (2\mathbb{Z}^3 - \mathbb{Z}^2c_2 + c_2^3 - 7\mathbb{Z}c_3 + 3c_4)e_n^4 + 0(e_n^5) \\ \Rightarrow 2 \frac{f(x_{n+1}^*)}{f'(x_n)} &= 2 \left\{ (\mathbb{Z} - c_2)e_n^2 + (2c_3 - \mathbb{Z}^2)e_n^3 + (\mathbb{Z}^3 - \mathbb{Z}^2c_2 + c_2^3 - 7\mathbb{Z}c_3 + 3c_4)e_n^4 + 0(e_n^5) \right\} \end{aligned}$$

$$= (2\mathbb{Z} - 2c_2)e_n^2 + (4c_3 - 2\mathbb{Z}^2)e_n^3 + (\mathbb{Z}^3 + 2c_2^3 - 14\mathbb{Z}c_3 + 6c_4)e_n^4 + 0(e_n^5) \quad (20)$$

Using equations (12), (15) and (18) we get

$$\begin{aligned} \frac{f(x_n)f(x_{n+1}^*)}{f'^2(x_n)} &= \frac{\{e_n + c_2e_n^2 + c_3e_n^3 + 0(e_n^4)\} \{-(c_2 - \mathbb{Z})e_n^2 + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 + 0(e_n^4)\}}{1 + (6\mathbb{Z}c_3 + 8c_4)e_n^3 + (6c_3 + \mathbb{Z}^2)e_n^2 + 2\mathbb{Z}e_n + (0)e_n^4} \\ &= \{e_n + c_2e_n^2 + c_3e_n^3 + 0(e_n^4)\} \{-(c_2 - \mathbb{Z})e_n^2 + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 + 0(e_n^4)\} \\ &\quad \{1 + (6\mathbb{Z}c_3 + 8c_4)e_n^3 + (6c_3 + \mathbb{Z}^2)e_n^2 + 2\mathbb{Z}e_n + (0)e_n^4\}^{-1} \\ &= (-c_2 + \mathbb{Z})e_n^3 + 0(e_n^4) \text{ after simplification} \\ \therefore \mathbb{Z} \frac{f(x_n)f(x_{n+1}^*)}{f'^2(x_n)} &= (\mathbb{Z}^2 - \mathbb{Z}c_2)e_n^3 + (0)e_n^4 \end{aligned} \quad (21)$$

Again, using equations (18), (19) and (15) we get

$$\begin{aligned} \frac{f(x_{n+1}^*)f'(x_{n+1}^*)}{f'^2(x_n)} &= \frac{\{-(c_2 - \mathbb{Z})e_n^2 + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 + 0(e_n^4)\} \{1 - [(\mathbb{Z}c_2 - \mathbb{Z}^2)e_n^2 + (\mathbb{Z}^2c_2 + 2\mathbb{Z}c_3 - \mathbb{Z}^3)e_n^3 + (0)e_n^4]\}}{[1 + 2\mathbb{Z}e_n + (6c_3 + \mathbb{Z}^2)e_n^2 + (6\mathbb{Z}c_3 + 8c_4)e_n^3 + (0)e_n^4]} \\ &= \{-(c_2 - \mathbb{Z})e_n^2 + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 + 0(e_n^4)\} \{1 - [(\mathbb{Z}c_2 - \mathbb{Z}^2)e_n^2 + (\mathbb{Z}^2c_2 + 2\mathbb{Z}c_3 - \mathbb{Z}^3)e_n^3 + (0)e_n^4]\} \\ &\quad \{[1 + 2\mathbb{Z}e_n + (6c_3 + \mathbb{Z}^2)e_n^2 + (6\mathbb{Z}c_3 + 8c_4)e_n^3 + (0)e_n^4]\}^{-1} \\ &= (\mathbb{Z} - c_2)e_n^2 + (\mathbb{Z}c_2 + 2c_3 - 2\mathbb{Z}^2)e_n^3 + (0)e_n^4 \text{ after simplification} \end{aligned} \quad (22)$$

Using equations (16), (20), (21) and (22) in (11) we obtain

$$\begin{aligned} x_{n+1} &= \alpha + (\mathbb{Z}c_2 + 2c_3 - \mathbb{Z}^2)e_n^3 - (c_2 - \mathbb{Z})e_n^2 - (4c_3 - 2\mathbb{Z}^2)e_n^3 - \mathbb{Z}e_n^2 - (\mathbb{Z}^2 - \mathbb{Z}c_2)e_n^3 + (\mathbb{Z} - c_2)e_n^2 \\ &\quad + (\mathbb{Z}c_2 + 2c_3 - 2\mathbb{Z}^2)e_n^3 + (0)e_n^4 \\ &= \alpha + (3\mathbb{Z}c_2 - 2\mathbb{Z}^2)e_n^3 + (-2c_2 + \mathbb{Z})e_n^2 + 0(e_n^4) \\ \Rightarrow x_{n+1} - \alpha &= (\mathbb{Z}c_2 - \mathbb{Z}^2)e_n^3 + 0(e_n^4) \end{aligned}$$

$$\therefore e_{n+1} = (\mathbb{Z}c_2 - \mathbb{Z}^2) e_n^3 + 0(e_n^4) \tag{23}$$

For $\mathbb{Z} = -1$ and 1 equation (23) shows that the new method has a third order rate of convergence,

Finally for $\mathbb{Z} = 0$, using equations (16), (20) and (21) in (11) we get

$$\begin{aligned} x_{n+1} = & \alpha - \left(-\mathbb{Z}^3 + \mathbb{Z}^2 c_2 + 5\mathbb{Z}c_3 - 3c_2c_3 - 3c_4 \right) e_n^4 - \left(\mathbb{Z}^2 - \mathbb{Z}c_2 - 2c_3 \right) e_n^3 - \left(-\mathbb{Z} + c_2 \right) e_n^2 \\ & - \left[\left(6\mathbb{Z}^3 - 4\mathbb{Z}^2 c_2 - 4\mathbb{Z}c_2^2 + 2c_2^3 - 20\mathbb{Z}c_3 + 12c_2c_3 + 6c_4 \right) e_n^4 + \left(-4\mathbb{Z}^2 + 4\mathbb{Z}c_2 + 4\mathbb{Z}c_3 \right) e_n^3 + \right. \\ & \left. + \left(2\mathbb{Z} - 2c_2 \right) e_n^2 \right] \\ & \left(7\mathbb{Z}^3 - 7\mathbb{Z}^2 c_2 - \mathbb{Z}c_2^2 + c_2^3 - 15\mathbb{Z}c_3 + 9c_2c_3 + 3c_4 \right) e_n^4 + \left(-3\mathbb{Z}^2 + 3\mathbb{Z}c_2 + 2c_3 \right) e_n^3 + \left(\mathbb{Z} - c_2 \right) e_n^2 \\ \Rightarrow e_{n+1} = & \left(2\mathbb{Z}^3 - 4\mathbb{Z}^2 c_2 + 3\mathbb{Z}c_3 - c_2c_3 \right) e_n^4 + 0(e_n^5) \tag{24} \end{aligned}$$

Equation (24) shows that for $\mathbb{Z} = 0$, the new method has a fourth order rate of convergence.

To demonstrate how effective the new method is, a comparison between the new method and other good and competent methods was conducted using fifty distinct problems. For the new method, we use the three schemes, Algorithms 2.1, 2.2 and 2.3. The comparison is based on the number of iterations that every method takes before the solution is reached. The methods used in the comparison are:

- i) Algorithms 2.1
- ii) Algorithms 2.2/[7]
- iii) Algorithms 2.3
- iv) [8]
- v) Newton Raphson's method
- vi) [9]
- vii) [10]

The complete fifty problems included in the analysis are in Appendix I, while the details of the comparison are in Appendix II.

4 RESULTS AND DISCUSSION

Statistical analysis was carried out from the numerical data to confirm the findings. A one-way ANOVA test was conducted and the following results were obtained taking into account, 95 percent confidence interval. As a result, we find a significant difference in the number of iterations between the methods analysed, since $p = 0.000 < 0.05$, see Table 1. Also from Table 2, since $p = 0.999 > 0.05$, for the solutions, we conclude that there is no significant difference in the average solutions obtained by the methods used.

Table 1: One-Way ANOVA Results for Number of Iterations to Convergence

	Sum of Squares	Df	Mean Square	F	Sig.
Between 3 Groups	185.295	6	30.882	14.656	0.000
Within Groups	672.178	319	2.107		
Total	857.472	325			

Df: degree of freedom; F: F distribution

Table 2: One-Way ANOVA Results for Solution values at Convergence

	Sum of Squares	Df	Mean Square	F	Sig.
Between Groups	2.339	6	0.390	0.050	0.999
Within Groups	2496.460	318	7.851		
Total	2498.799	324			

The complete results of the number of iterations obtained for all the tested methods across all the 50 problems are in Appendix II. To analyse the difference in the number of iterations, we used the post Hoc Test (Duncan Multiple range) and from the results obtained, which are in Table 3, we see that Algorithm 2.2/CM, BM, Algorithm 2.1 and Algorithm 2.3 have the least number of iterations all in the first homogeneous subset. Then, NSM together with Newton Raphson method are in the second homogeneous subset. Finally, in the third homogeneous subset we have NM, having the highest number of iterations.

Note that, Degree of freedom is the number of independent ways by which a dynamic system can move, without violating any constraint imposed on it. F distribution is a continuous probability distribution that arise frequently as the null distribution of a test statistic. For more explanation on ANOVA and interpretations of its results see [14] and [15].

Table 3: Homogeneous Subsets (Post Hoc Test) showing Number Iterations to Convergence in respect of the methods

Methods	N	Subset for alpha = 0.05		
		1	2	3
Algorithm2.2/CM	50	2.70		
BM	45	2.89		
Algorithm 2.1	50	3.02		
Algorithm 2.3	50	3.10		
Newton Raphson	46		3.83	
NSM	45		4.29	
NM	40			4.95
Sig.		0.232	0.126	1.000

N: Number of successful events (number of problems in Appendix I for which the solution converge)

Table 4: Descriptive Statistics for each method in respect of Number of Iterations to Convergence

	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean		Min	Max
					Lower Bound	Upper Bound		
NM	40	4.95	1.797	0.284	4.38	5.52	2	9
Newton Raphson	46	3.83	1.805	0.266	3.29	4.36	1	11
Algorithm2.2/CM	50	2.70	1.216	0.172	2.35	3.05	1	7
NSM	45	4.29	1.753	0.261	3.76	4.82	2	11
BM	45	2.89	0.859	0.128	2.63	3.15	2	5
Algorithm 2.3	50	3.10	1.359	0.192	2.71	3.49	1	8
Algorithm 2.1	50	3.02	1.186	0.168	2.68	3.36	1	7
Total	326	3.49	1.624	0.090	3.31	3.67	1	11

As can be seen from Table 4, the descriptive statistics shows the mean, standard deviation and standard error of the number of iterations to convergence in respect of each of the methods. Algorithm 2.1, Algorithm 2.2/ CM and Algorithm 2.3 converge to all the fifty solutions, while, four and five of the solutions did not converge in the case of Newton Raphson and BM respectively. Algorithm 2.2/ CM has the smallest mean of 2.70, followed by BM and Algorithm 2.1 with 2.89 and 3.02 respectively. In the case of standard deviation, BM has the best standard deviation of 0.859 with standard error of 0.128 followed by Algorithm 2.1 with standard deviation of 1.186 and standard error of 0.168. Algorithm 2.2/ CM has a standard deviation of 1.216 with standard error of 0.172. Finally, Algorithm 2.3 has a standard deviation of 1.359 with standard error of 0.192.

5 CONCLUSION

In this paper, we present a family of iterative methods for solving nonlinear equations. The new methods are compared with five other existing methods in terms of number of iterations to convergence. Fifty problems are used in the comparison are given in Appendix I, while the complete results of the number of iterations in respect of each method are in Appendix II. From the analysis of variance obtained, there is a significant difference between the number of iterations to convergence obtained from the methods used. The solutions of the three new algorithms converge in all the fifty problems as compared to others, whose solutions did not converge in some of the problems. In the case of standard deviation and standard error, BM has the best, with a standard deviation of 0.895 and a standard error of 0.128. Algorithm 2.1 and Algorithm 2.2/CM come next with standard deviations of 1.186 and 1.216 respectively having standard errors 0.168 and 0.172 respectively. Thus the New methods are comparatively good in all the aspects being considered. Among the three new algorithms, algorithms 2.1 and 2.2 give the results. Therefore, any of the two new algorithms can be considered as an alternative method for solving nonlinear equations.

ACKNOWLEDGEMENT

The authors are grateful to the reviewers for their suggestions and corrections.

REFERENCES

- [1] G. Adomian, *Nonlinear Stochastic Operator Equations*. Orlando, Florida: Academic Press, 1986.
- [2] V. Daftardar-Gejji and H. Jafari, "An iterative method for solving nonlinear functional equations," *Journal of Mathematical Analysis and Applications*, vol. 316, pp. 753–763, 2006.
- [3] M. A. Noor and K. I. Noor, "Three step iterative methods for nonlinear equations," *Journal of Applied Mathematics and Computational*, vol. 183, pp. 322-327, 2006.
- [4] J. H. Yun, "A note on three step iterative method for nonlinear equations," *Journal of Applied Mathematics and Computational*, vol. 202, pp. 401-405, 2008.
- [5] J. H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, pp. 257–262, 1999.
- [6] M. A. Noor and W.A. Khan, "New iterative methods for solving nonlinear equation by using homotopy perturbation method", *Applied Mathematics and Computational*, vol. 219 pp. 3565-3574, 2012.
- [7] C. Chun, "Iterative methods improving Newton's method by the decomposition method," *Applied Mathematics and Computation*, vol. 50, pp. 1559–1568, 2005.
- [8] M. A. Noor, "New family of iterative methods for nonlinear equations," *Applied Mathematics and Computational*, vol. 190, pp. 553-558, 2007.

- [9] M. Basto, V. Semiao, and F. Calheiros, "A new iterative method to compute nonlinear equations." *Applied Mathematics and Computation*, vol. 173 pp. 468–483, 2006.
- [10] M. S. Ndayawo and B. Sani, "New iterative schemes for solving nonlinear equations," *Journal of the Nigeria Association of Mathematical Physics*, vol. 25, no.1, pp. 425-438, 2013.
- [11] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*. Boston: Kluwer Academic Publishers, 1994.
- [12] R. Rach, "A convenient form for the adomian polynomials," *Journal of Mathematical Analysis and Applications*, vol. 102, pp. 415-419, 1984.
- [13] R. L. Burden and J. D. Faires, *Numerical Analysis*, 9th ed. California: Brooks/Cole, 2011.
- [14] R. G. Miller, *Beyond ANOVA, Basics of Applied Statistics* in Wiley Series in Probability and Statistics. New York: John Wiley & Sons, Inc., 1986.
- [15] M. S. Paoella, *Linear Models and Time-Series Analysis: Regression, ANOVA, ARMA and GARCH* in Wiley Series in Probability and Statistics. Hoboken, NJ: John Wiley & Sons Ltd, 2019.
- [16] C. Chun and B. Neta, "Some modification of Newton method by the method of undetermined coefficients," *Journal of Computers and Mathematics with Applications*, vol. 56, pp. 2528 -2538, 2008.
- [17] M. Kumar, A. K. Singh, and A. Srivastava, "Various Newton-type iterative methods for solving nonlinear equations," *Journal of the Egyptian Mathematical Society*, vol. 21 pp, 334-339, 2013.
- [18] F. Soleymani, "Novel computational iterative methods with optimal order for nonlinear equations," *Advances in Numerical Analysis*, vol. 2011, ID 270903, 2011.

APPENDIX I: FIFTY PROBLEMS USED IN THE COMPARISON

1. $\sin^2 x - x^2 + 1 = 0$

2. $x^2 - e^x - 3x + 2 = 0$

3. $\cos x - x = 0$

4. $(x-1)^3 - 1 = 0$

5. $x^3 - 10 = 0$

6. $xe^{x^2} - \sin^2 x + 3\cos x - x = 0$

7. $xe^{x^2} - \sin^2 x + 3\cos x + 5 = 0$

8. $e^{x+7x-30} - 1 = 0$

9. $e^{2x} + 2x + 0.1 = 0$

10. $2x^3 - x^2 - 7x + 6 = 0$

11. $x^3 - 9x + 1 = 0$

12. $x^3 - 3x - 1.06 = 0$

13. $x^3 - 6x + 4 = 0$

14. $2x - 3\sin x - 5 = 0$

15. $x^3 - 3x + 1 = 0$

16. $3x - \ln x - 16 = 0$

17. $\cos x - 2x + 3 = 0$

18. $x + \ln x - 2 = 0$

19. $x^3 - x - 0.1 = 0$

20. $x^4 - 12x + 7 = 0$

21. $x + \sin x + 0.5 = 0$

22. $x - 2 - e^{-x} = 0$

23. $x^3 + 4x^2 + 8x + 8 = 0$

24. $2x - \ln x - 7 = 0$

25. $x^2 - 1.25x + 0.25 = 0$

26. $x^2 + 5.15x + 2.37 = 0$

27. $x^4 - 11x + 8 = 0$

28. $\ln x - x + 3 = 0$

29. $e^x - x - 3 = 0$

30. $2x^2 - 12x + 11 = 0$

31. $x^4 - 5x^2 + x + 3 = 0$

32. $x^3 + 4x - 9 = 0$

33. $x^7 - 3x^6 + 3x^5 - 13x^4 + 43x^3 - 57x^2 + 33x - 7 = 0$

34. $\sin x - \frac{x}{2} = 0$

35. $x^5 + x - 1000 = 0$

36. $\sqrt{x} - \frac{1}{x} - 3 = 0$

37. $\ln x + \sqrt{x} - 5 = 0$

38. $e^x \sin x - 2x - 5 = 0$

39. $e^{-x} - \cos x = 0$

40. $\cos^2 x - \frac{x}{5} = 0$

41. $(1 + \cos x)(e^x - 2) = 0$

42. $e^{-x} + \sin x - 1 = 0$

43. $xe^{-x} - 0.1 = 0$

44. $x^2 + \sin x + x = 0$

45. $\sin(2 \cos x) - 1 - x^2 + e^{\sin^3 x} = 0$

46. $x^6 - 10x^3 + x^2 - x + 3 = 0$

47. $x^4 - x^3 + 11x - 7 = 0$

48. $x^3 - \cos x + 2 = 0$

49. $\sqrt{x} - \cos x = 0$

50. $\ln x - x^3 + 2 \sin x = 0$

Problems 1-8 are from [8], problems 9-34 are from [100], problems 35-38 are from [16] problems 39-42 are from [17] and problems 42-50 are from [18].

APPENDIX II: COMPARISON BETWEEN NUMBER OF ITERATIONS

	Initial point (x_0)	x_n (Solutions)	NM	NR	Algo 2.2/ CM	BM	NSM	Algo 2.1	Algo 2.3
1	1.3	1.404491649	4	4	3	2	3	3	3
2	2	0.257530285	9	5	3	5	6	4	4
3	1.7	0.739085133	5	4	3	3	6	4	3
4	3.5	2	6	6	4	3	5	4	3
5	1.5	2.15443469	5	5	4	3	4	3	4
6	-2	-0.915306601	5	4	6	DIV	DIV	5	6
7	-2	-1.207647827	5	DIV	5	DIV	4	4	5
8	3.5	3	6	DIV	7	DIV	11	7	7
9	-0.05	-0.31584581	4	4	2	3	3	3	3
10	0.86	1	5	4	5	3	4	2	3
11	0.11	0.111264158	2	2	1	2	2	2	1
12	-0.35	-0.370252219	3	3	2	4	2	2	2
13	0.67	0.732050808	3	4	2	2	3	2	4
14	2.5	2.883236873	DIV	3	2	3	4	3	3
15	0.33	0.347296355	3	3	2	2	3	2	3
16	5.3	5.926476625	DIV	2	2	2	4	2	2
17	1.5	1.523592933	5	3	1	2	3	2	2
18	2	1.557145599	7	3	3	3	5	3	3
19	-0.1	-0.101031258	2	2	1	2	2	2	2
20	0.58	0.59368584	3	4	2	3	3	2	2
21	-0.5	-0.251318625	7	3	2	3	4	3	3
22	2	2.120028239	5	3	2	2	3	2	2
23	-1	-2	5	1	1	4	7	1	1
24	3.5	4.219906484	DIV	3	2	2	4	3	3
25	0.2	0.25	DIV	3	2	3	4	2	2
26	-0.2	-0.510871855	4	4	3	2	4	3	3
27	0.73	0.757149516	3	3	2	2	4	2	2
28	3	4.505241496	DIV	5	2	3	7	3	3
29	-3	-2.947530903	6	2	2	2	3	2	2
30	0.92	1.129171307	5	4	3	3	5	2	3
31	-3	-2.198691243	7	6	4	4	6	4	4
32	2.25	1.464595701	4	5	3	3	4	4	3
33	0.21	0.593685833	DIV	7	4	3	8	6	5
34	2	1.895494267	3	3	2	2	3	2	2
35	4	3.977899394	3	3	2	2	2	3	2
36	1	9.633595562	DIV	5	4	DIV	DIV	4	8
37	3	8.309432693	DIV	5	3	5	DIV	6	5
38	-1	-2.52324523	9	3	2	3	6	3	3
39	1.5	1.29269572	4	4	2	3	5	3	3
40	0.5	1.085982678	4	4	2	3	4	3	3
41	0.9	0.693147181	7	3	2	3	4	2	2
42	2.3	2.076831274	4	4	3	2	3	3	3
43	-0.1	0.111832559	5	4	3	3	5	3	3
44	0.3	0.00E+00	DIV	5	3	DIV	DIV	4	4
45	-0.82	0.8135737292	3	3	2	2	4	2	2
46	0.81	0.6586048471	6	DIV	3	3	4	3	2
47	0.9	0.6450239555	5	3	3	4	5	4	4
48	-1.1	-1.172577964	4	4	3	3	4	2	3

49	1.3	0.6417143709	DIV	4	2	3	6	3	3
50	1.4	1.297997743	4	DIV	2	4	DIV	3	2

DIV: Diverging

NR: Newton Raphson