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# Integrals Involving the Homogeneous Generalized Hypergeometric Function and Multivariable Aleph-Function

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**ABSTRACT**

*Recently, T two finite integrals involving the homogeneous hypergeometric function and the modified multivariable H function have been evaluated. During this work, we established two unified finite integrals whose integrands are the product of the multivariable Aleph function, the homogeneous generalized hypergeometric function and the Gegenbauer polynomial. Several corollaries and remarks will be cited at the end of our study.*

**Keywords:** Multivariable Aleph-function, homogeneous generalized hypergeometric function, multiple Mellin-Barnes contour integrals, Gegenbauer polynomial, multivariable H-function, Aleph-function of two variables, I-function of two variables, Aleph-function of one variable, I- function of one variable.
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## 1 INTRODUCTION

Gour and Singh [1] are interested in two finite integrals involving the homogeneous hypergeometric function [2], the modified multivariable H-function [3] and the Gegenbauer polynomial [4, CH.17]. From this work, the authors generalize the integrals cited above, replacing the modified multivariable H-function by the multivariable Aleph- function.

First, we define these two functions mentioned above. In the second section, we will see two required results concerning the Gegenbauer polynomial. The fundamental results will be given in the third section. In the last section, we will see several special cases and observations.

The homogeneous generalized hypergeometric function ${}_pB_q\left(\alpha_p, \beta_q; z\right)$ introduced by Basister [2]:

$${}_pB_q\left(\alpha_p, \beta_q; z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \Omega\left(\alpha_{p+n}, \beta_{q+n}, 0\right) \frac{z^n}{n!} \tag{1.1}$$

where $\Omega(.)$ is generalized modified Struve function ([2], p. 96) defined as

$$\Omega(a,c,z)=\frac{1}{2^{1+c}\pi^2 e^{\omega\pi c}}\,\Gamma(1-a)\Gamma(c)\Gamma(1+a-c)$$
$$\times e^{\left(\frac{z}{2}\right)}\Bigg[(1-e^{2\omega\pi a})\int_0^{(1+)} e^{\left(\frac{zu}{2}\right)}(1+u)^{\alpha-1}(1-u)^{c-a-1}\,du$$
$$+\left(1-e^{2\omega\pi a(c-a)}\right)\int_0^{(-1+)} e^{\left(\frac{zu}{2}\right)}(1+u)^{\alpha-1}(1-u)^{c-a-1}\,du\Bigg] \qquad (1.2)$$

where $\omega=\sqrt{-1}$, $\Re(\beta_q)>\Re(\beta_p)$, the series (1.1) converges for all z if $p\le q$; series converges for |z| < 1 if $p = q+1$; it diverges if $p > q+1$. Here we impose $2\le p\le q+1$.

The multivariable Aleph-function [5] is a unified special function of several variables. It generalizes the multivariable I-function [6], which itself is an extension of the multivariable H-function [7,8], see the recent papers [9] and [10]. In passing, we note that the multivariable Aleph-function is a unification of the multivariable H-function and the Aleph-function of one variable [11]. Its integral representation is:

$$\aleph(z_1,z_2,\dots,z_r)=\aleph^{0,n:V}_{p_i,q_i,\tau_i;R;W}\left(\begin{matrix}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{matrix}\left|\begin{matrix}A:C\\ \\ \\ \\ B:D\end{matrix}\right.\right)=\frac{1}{(2\pi\omega)^r}\int_{L_1}\dots\int_{L_r}\psi(\xi_1,\dots,\xi_r)\prod_{i=1}^{r}\left(\phi_i(\xi_i)z_i^{\xi_i}\right)d\xi_1\dots d\xi_r$$

(1.3)

where $\omega=\sqrt{-1}$ ,

$V=m_1n_1;m_2n_2;\dots;m_rn_r$

$W=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\dots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}$

$A=\left[\left(a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right)_{1,n}\right],\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\dots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right]$

$C=\left[\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1}\right],\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}}\right];\dots;\left[\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_{i^{(r)}}}\right]$

$B=\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\dots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right]$

$D$
$=\left[\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,m_1}\right],\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}}\right];\dots;\left[\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_{i^{(r)}}}\right]$

and

$$\psi(\xi_1,\dots,\xi_r)=\frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}\xi_k\right)}{\sum_{i=1}^{R}\left[\tau_i\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}\xi_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}\xi_k\right)\right]} \tag{1.4}$$

$$\phi_k(\xi_k)=\frac{\prod_{j=1}^{m_k}\Gamma\left(d_j^{(k)}-\delta_j^{(k)}\xi_k\right)\prod_{j=1}^{n_k}\Gamma\left(1-c_j^{(k)}+\gamma_j^{(k)}\xi_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma\left(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}\xi_k\right)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma\left(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}\xi_k\right)\right]} \tag{1.5}$$

the parameters $a_j(j=1,\dots,p), b_j(j=1,\dots,q), c_j^{(k)}(j=1,\dots,n_k), c_{ji^{(k)}}^{(k)}(j=n_k+1,\dots,p_{i^{(k)}}),\ d_j^{(k)}(j=1,\dots,m_k), d_{ji^{(k)}}^{(k)}(j=m_k+1,\dots,q_{i^{(k)}})$ $(k=1,\dots,r; i=1,\dots,R$ and $i^{(k)}=1,\dots,R^{(k)})$ are complex numbers. Also $\alpha$'s, $\beta$'s, $\gamma$'s and $\delta$'s is assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)}=\sum_{j=1}^{n}\alpha_j^{(k)}+\tau_i\sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}+\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}-\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}\leq 0 \tag{1.6}$$

The real numbers $\tau_i>0\ (i=1,\dots,R)$ and $\tau_{i^{(k)}}>0\ (i=1,\dots,R)$.

The contour is in the $s_k$-plane and run from $\sigma-\omega\infty$ to $\sigma+\omega\infty$, where $\sigma$ is a real number with loop , if necessary, ensure that the poles of $\Gamma\left(d_j^{(k)}-\delta_j^{(k)}\xi_k\right)$ with $j=1,\dots,m_k$ are separated from those of $\Gamma\left(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}\xi_k\right)$ with $j=1,\dots,n$ and $\Gamma\left(1-c_j^{(k)}+\gamma_j^{(k)}\xi_k\right)$ with $j=1,\dots,n_k$ to the left of the contour $L_k$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained by extension of the corresponding conditions for multivariable H-function as:

$$|\arg z_i|<\frac{1}{2}A_i^{(k)}\pi$$

where

$$A_i^{(k)}=\sum_{j=1}^{n}\alpha_j^{(k)}-\tau_i\sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0 \tag{1.7}$$

with $k=1,\dots,r; i=1,\dots,R$ and $i^{(k)}=1,\dots,R^{(k)}$. For more details, one can refer [1].

## 2 REQUIRED RESULTS

For the Gegenbauer polynomial [4, Ch.17], we use the integral ([12], Eq.4, p.1221) and take $x=1-2y^2$ , we get this relation.

**Lemma 1**.

$$\int_0^1 \boldsymbol{C_m^\lambda}\left(\mathbf{1-2y^2}\right)\left(\mathbf{1-y^2}\right)^{\lambda-\frac{1}{2}}\boldsymbol{y^{2\lambda+2u+2v}dy} = \frac{\sqrt{\pi}}{\mathbf{4^\lambda\Gamma(\lambda)}}\frac{(-\mathbf{1})^m\Gamma(\boldsymbol{m}+\mathbf{2}\lambda)}{\boldsymbol{m}!}\frac{\Gamma(\boldsymbol{u+v}+\mathbf{1})\Gamma\left(\boldsymbol{u+v}+\lambda+\frac{1}{2}\right)}{\Gamma(\boldsymbol{u+v-m}+\mathbf{1})\Gamma(\boldsymbol{u+v+m}+\mathbf{2}\lambda+\mathbf{1})} \tag{2.1}$$

where

$$\lambda+u+v>-\frac{1}{2}, \boldsymbol{C_m^\lambda}(x)=\frac{(2v)_m P_m^{\left(v-\frac{1}{2},v-\frac{1}{2}\right)}(x)}{\left(v+\frac{1}{2}\right)}$$

and $P_m^{(\alpha,\alpha)}(.)$ is the Jacobi polynomial.

**Lemma 2** ([4], Eq.1, p.276).

$$(1-2xh+h^2)^{-\lambda}=\sum_{m=0}^{\infty} C_m^\lambda(x)h^m \tag{2.2}$$

## 3 MAIN INTEGRAL FORMULAS

In this part, we determine the expressions of two unified finite integrals. For that, we will apply the results of the previous section.

**Theorem 1.**

If $\sigma_i>0 (i=1,\dots,r), \Re(\beta_q)>\Re(\alpha_p)$ for $0\le p\le q+1, \Re(\lambda+\mu)+\sum_{i=1}^r \alpha_i \min_{1\le j\le m_i} \Re\left[\frac{d_j^{(i)}}{\delta_j^{(i)}}\right]>-\frac{1}{2}$ and $|\arg z_i|<\frac{1}{2}A_i^{(k)}\pi$ then the following formula holds

$$\int_0^1 C_m^\lambda(1-2y^2)(1-y^2)^{\lambda-\frac{1}{2}}y^{2\lambda+2u}{}_pB_q(\alpha_s,\beta_t:(zy)^2)\aleph_{p_i,q_i,\tau_i;R;W}^{0,n:V}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\ \cdot\\ \cdot\\ \cdot\\ (xy^2)^{\sigma_r}\end{matrix}\middle| \begin{matrix}A:C\\ \\ \\ \\ B:D\end{matrix}\right\}dy$$

$$=\frac{\sqrt{\pi}}{4^\lambda\Gamma(\lambda)}\frac{(-1)^m\Gamma(m+2\lambda)}{m!}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\dots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{z^{2k}}{k!}$$

$$\times\aleph_{p_i+2,q_i+2,\tau_i;R;W}^{0,n+2:V}\left\{\begin{matrix}x^{\sigma_1}\\ \cdot\\ \cdot\\ \cdot\\ x^{\sigma_r}\end{matrix}\middle| \begin{matrix}(-k-u:\sigma_1,\dots,\sigma_r),\left(\frac{1}{2}-k-u-\lambda:\sigma_1,\dots,\sigma_r\right),A:C\\ \\ \\ \\ B,(-k-u+m:\sigma_1,\dots,\sigma_r),(-k-u-m-2\lambda:\sigma_1,\dots,\sigma_r):D\end{matrix}\right\} \tag{3.1}$$

**Theorem 2.**

If $\sigma_i > 0 (i = 1, \dots, r), \Re(\beta_q) > \Re(\alpha_p)$ for $0 \le p \le q+1, \Re(u) + \sum_{i=1}^{r} \alpha_i \min_{1\le j\le m_i} \Re\left[\frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > -\frac{1}{2}$ and $|\arg z_i| < \frac{1}{2} A_i^{(k)} \pi$ then the following formula holds

$$\int_0^1 (1-y^2)^{\lambda-\frac{1}{2}} y^{2u} \, {}_pB_q(\alpha_s, \beta_t : (zy)^2) \aleph^{0,n:V}_{p_i,q_i,\tau_i;R;W} \left\{ \begin{matrix} (xy^2)^{\sigma_1} \\ \cdot \\ \cdot \\ \cdot \\ (xy^2)^{\sigma_r} \end{matrix} \middle| \begin{matrix} A:C \\ \\ \\ \\ B:D \end{matrix} \right\} dy$$

$$= \frac{\sqrt{\pi}}{\Gamma(\lambda)} \frac{(-1)^m \Gamma(m+2\lambda)}{m!} \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \Omega(\alpha_{p+k}, \beta_{q+k}, 0) \frac{z^{2k}}{k!}$$

$$\times \aleph^{0,n+2:V}_{p_i+2,q_i+2,\tau_i;R;W} \left\{ \begin{matrix} x^{\sigma_1} \\ \cdot \\ \cdot \\ \cdot \\ x^{\sigma_r} \end{matrix} \middle| \begin{matrix} (-k-u:\sigma_1,\dots,\sigma_r), \left(\frac{1}{2}-k-u-\lambda:\sigma_1,\dots,\sigma_r\right), A:C \\ \\ \\ \\ B, (-k-u+m:\sigma_1,\dots,\sigma_r), (-k-u-m-2\lambda:\sigma_1,\dots,\sigma_r):D \end{matrix} \right\} \quad (3.2)$$

**Proof of Theorem 1 and 2:** To prove the first theorem, On the left-hand side of (3.1), we apply the definition of the homogeneous hypergeometric function ${}_pB_q(\alpha_p, \beta_q; z)$ by (1.1), expressing the multivariable Aleph-function with the help of the integral representation defined by (1.3) and then interchange the order of integrations and summations, (possible under the conditions set), we can write (say I):

$$I = \int_0^1 \boldsymbol{C_m^\lambda}(1-2y^2)(1-y^2)^{\lambda-\frac{1}{2}} y^{2\lambda+2u} \, {}_pB_q(\alpha_s, \beta_t : (zy)^2) \aleph^{0,n:V}_{p_i,q_i,\tau_i;R;W} \left\{ \begin{matrix} (xy^2)^{\sigma_1} \\ \cdot \\ \cdot \\ \cdot \\ (xy^2)^{\sigma_r} \end{matrix} \middle| \begin{matrix} A:C \\ \\ \\ \\ B:D \end{matrix} \right\} dy$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \Omega(\alpha_{p+k}, \beta_{q+k}, 0) \frac{z^{2k}}{k!} \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(\xi_1, \dots, \xi_r) \prod_{i=1}^{r} \left(\phi_i(\xi_i) z_i^{\xi_i}\right)$$

$$\times \left\{ \int_0^1 C_m^\lambda (1-2y^2)(1-y^2)^{\lambda-\frac{1}{2}} y^{2\lambda+2u+2k+2\sum_{i=1}^{r}\sigma_i\xi_i} dy \right\} d\xi_1 \dots d\xi_r \quad (3.3)$$

we calculate the inner integral by applying the lemma 1 and acquire following:

$$I = \frac{\sqrt{\pi}}{\Gamma(\lambda)} \frac{(-1)^m \Gamma(m+2\lambda)}{m!} \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \Omega(\alpha_{p+k}, \beta_{q+k}, 0) \frac{z^{2k}}{k!}$$

$$\times \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\prod_{i=1}^{r}\left(\phi_i(\xi_i)z_i^{\xi_i}\right)$$

$$\times\left\{\frac{\Gamma(u+k+\sum_{i=1}^{r}\sigma_i\xi_i+1)\Gamma\left(u+k+\sum_{i=1}^{r}\sigma_i\xi_i+\lambda+\frac{1}{2}\right)}{\Gamma(u+k+\sum_{i=1}^{r}\sigma_i\xi_i-m+1)\Gamma(u+k+\sum_{i=1}^{r}\sigma_i\xi_i+m+2\lambda+1)}\right\}d\xi_1\ldots d\xi_r \quad (3.4)$$

Now, using the multiple integrals contour as an Aleph function of several complex variables, the equation (3.1) is obtained. To prove the second theorem, we use the lemma 2, by putting $x = 1-2y^2$, we have the relation:

$$(4y^2)^{-\lambda} = \sum_{m=0}^{\infty} C_m^{\lambda}(1-2y^2) \quad (3.5)$$

multiplying both sides of (3.5) by

$$(1-y^2)^{\lambda-\frac{1}{2}}y^{2\lambda+2u}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2)\aleph^{0,n:V}_{p_i,q_i,\tau_i;R;W}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\ \cdot\\ \cdot\\ \cdot\\ (xy^2)^{\sigma_r}\end{matrix}\middle|\begin{matrix}A:C\\ \\ \\ \\ B:D\end{matrix}\right\} \quad (3.6)$$

and integrating with respect to $y$ between 0 to 1, this gives:

$$4^{-\lambda}\int_{\mathbf{0}}^{\mathbf{1}}(1-y^2)^{\lambda-\frac{1}{2}}y^{2u}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2)\aleph^{0,n:V}_{p_i,q_i,\tau_i;R;W}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\ \cdot\\ \cdot\\ \cdot\\ (xy^2)^{\sigma_r}\end{matrix}\middle|\begin{matrix}A:C\\ \\ \\ \\ B:D\end{matrix}\right\}dy$$

$$= \int_{\mathbf{0}}^{\mathbf{1}}(1-y^2)^{\lambda-\frac{1}{2}}y^{2u+2\lambda}\sum_{m=0}^{\infty}C_m^{\lambda}(1-2y^2)\,{}_pB_q(\alpha_s,\beta_t:(zy)^2)\aleph^{0,n:V}_{p_i,q_i,\tau_i;R;W}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\ \cdot\\ \cdot\\ \cdot\\ (xy^2)^{\sigma_r}\end{matrix}\middle|\begin{matrix}A:C\\ \\ \\ \\ B:D\end{matrix}\right\}dy \quad (3.7)$$

we permute the order of integral and summation (this is permissible according to the conditions imposed), we have (say J):

$$J = \sum_{m=0}^{\infty}\int_0^1 C_m^{\lambda}(1-2y^2)\,(1-y^2)^{\lambda-\frac{1}{2}}y^{2u+2\lambda}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2)\aleph^{0,n:V}_{p_i,q_i,\tau_i;R;W}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\ \cdot\\ \cdot\\ \cdot\\ (xy^2)^{\sigma_r}\end{matrix}\middle|\begin{matrix}A:C\\ \\ \\ \\ B:D\end{matrix}\right\}dy \quad (3.8)$$

and using the formula (3.1), we get the expression (3.2).

## 4 SPECIAL CASES

In this section, we give some cases and remarks. We will use the Aleph-function of two variables defined by Kumar [13] and Sharma [14], the I-function of two variables studied by Sharma and Mishra [15], the multivariable H-function due to Srivastava and Panda [7,8], the Aleph-function of one variable described by Südland et al. [11] and the I-function introduced by Saxena [16].

(i) If $r=2$, the multivariable Aleph-function reduces to Aleph-function of two variables. We note for convenience:

$$A_2=\left[\left(a_j;\alpha_j,A_j\right)_{1,n_1}\right],\left[\tau_i\left(a_{ji};\alpha_{ji},A_{ji}\right)_{n_1+1,p_i}\right];B_2=\left[\tau_i\left(b_{ji};\beta_{ji},B_{ji}\right)_{1,q_i}\right]$$

$$C_2=\left[\left(c_j,\gamma_j\right)_{1,n_2}\right],\left[\tau_{i'}\left(c_{ji'},\gamma_{ji'}\right)_{n_2+1,p_{i'}}\right];\left[\left(e_j,E_j\right)_{1,n_3}\right],\left[\tau_{i''}\left(e_{ji''},\gamma_{ji''}\right)_{n_3+1,p_{i''}}\right]$$

$$D_2=\left[\left(d_j,\delta_j\right)_{1,m_2}\right],\left[\tau_{i'}\left(d_{ji'},\delta_{ji'}\right)_{m_2+1,q_{i'}}\right];\left[\left(f_j,F_j\right)_{1,m_3}\right],\left[\tau_{i''}\left(f_{ji''},F_{ji'}\right)_{m_3+1,q_{i''}}\right]$$

We obtain easily the integrals involving the Aleph-function of two variables by using the Theorem 1 as

**Corollary 1**

$$\int_{\mathbf{0}}^{\mathbf{1}} \boldsymbol{C}_{\boldsymbol{m}}^{\boldsymbol{\lambda}}(1-2y^2)\,(1-y^2)^{\lambda-\frac{1}{2}}y^{2\lambda+2u}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2)\aleph_{p_i,q_i,\tau_i;R;W}^{0,n:V}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\(xy^2)^{\sigma_2}\end{matrix}\middle|\begin{matrix}A_2:C_2\\B_2:D_2\end{matrix}\right\}dy$$

$$\aleph_{p_i+2,q_i+2,\tau_i;r;p_{i'},q_{i'},\tau_{i'};r';p_{i''},q_{i''},\tau_{i''};r''}^{0,n_1+2;m_2,n_2;m_3,n_3}\left\{\begin{matrix}x^{\sigma_1}\\x^{\sigma_2}\end{matrix}\middle|\begin{matrix}(-k-u:\sigma_1,\sigma_2),\left(\frac{1}{2}-k-u-\lambda:\sigma_1,\sigma_2\right),A_2:C_2\\B_2,(-k-u+m:\sigma_1,\sigma_2),(-k-u-m-2\lambda:\sigma_1,\sigma_2):D_2\end{matrix}\right\}$$
(4.1)

while respecting the conditions mentioned in (3.1) with $r=2$.

We have the second integral involving the Aleph-function of two variables by applying the Theorem 2.

**Corollary 2**

$$\int_{\mathbf{0}}^{\mathbf{1}}(1-y^2)^{\lambda-\frac{1}{2}}y^{2u}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2)\aleph_{p_i,q_i,\tau_i;R;W}^{0,n:V}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\(xy^2)^{\sigma_2}\end{matrix}\middle|\begin{matrix}A_2:C_2\\B_2:D_2\end{matrix}\right\}dy$$

$$=\frac{\sqrt{\pi}}{\Gamma(\lambda)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+2\lambda)}{m!}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\Omega\left(\alpha_{p+k},\beta_{q+k},0\right)\frac{z^{2k}}{k!}$$

$$\times \aleph^{0,n_1+2;m_2,n_2;m_3,n_3}_{p_i+2,q_i+2,\tau_i;r;p_{i'},q_{i'},\tau_{i'};r';p_{i''},q_{i''},\tau_{i''};r''}\left\{\begin{matrix} x^{\sigma_1} \\ x^{\sigma_2}\end{matrix}\middle| \begin{matrix} (-k-u:\sigma_1,\sigma_2),\left(\frac{1}{2}-k-u-\lambda:\sigma_1,\sigma_2\right),A_2:C_2 \\ B_2,(-k-u+m:\sigma_1,\sigma_2),(-k-u-m-2\lambda:\sigma_1,\sigma_2):D_2\end{matrix}\right\}$$
(4.2)

valid while respecting the conditions given in (3.2) with $r=2$.

(ii) If $\tau_i,\tau_{i'},\tau_{i''}\to 1$, we obtain the I-function of two variables, we pose to simplify the expressions:

$$A_3=\left[(a_j;\alpha_j,A_j)_{1,n_1}\right],\left[(a_{ji};\alpha_{ji},A_{ji})_{n_1+1,p_i}\right];B_3=\left[(b_{ji};\beta_{ji},B_{ji})_{1,q_i}\right]$$

$$C_3=\left[(c_j,\gamma_j)_{1,n_2}\right],\left[\left(c_{ji'},\gamma_{ji'}\right)_{n_2+1,p_{i'}}\right];\left[(e_j,E_j)_{1,n_3}\right],\left[\left(e_{ji''},\gamma_{ji''}\right)_{n_3+1,p_{i''}}\right]$$

$$D_3=\left[(d_j,\delta_j)_{1,m_2}\right],\left[\left(d_{ji'},\delta_{ji'}\right)_{m_2+1,q_{i'}}\right];\left[(f_j,F_j)_{1,m_3}\right],\left[\left(f_{ji''},F_{ji'}\right)_{m_3+1,q_{i''}}\right]$$

and we get the integrals about the I-function of two variables as:

**Corollary 3**

$$\int_0^1 \boldsymbol{C}^{\lambda}_{\boldsymbol{m}}(1-2y^2)\,(1-y^2)^{\lambda-\frac{1}{2}}y^{2\lambda+2u}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2)I^{0,n:V}_{p_i,q_i;R;W}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\(xy^2)^{\sigma_2}\end{matrix}\middle|\begin{matrix}A_3:C_3\\B_3:D_3\end{matrix}\right\}dy$$

$$=\frac{\sqrt{\pi}}{4^{\lambda}\Gamma(\lambda)}\frac{(-1)^m\Gamma(m+2\lambda)}{m!}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\dots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{z^{2k}}{k!}$$

$$\times I^{0,n_1+2;m_2,n_2;m_3,n_3}_{p_i+2,q_i+2,r;p_{i'},q_{i'};r';p_{i''},q_{i''};r''}\left\{\begin{matrix} x^{\sigma_1} \\ x^{\sigma_2}\end{matrix}\middle| \begin{matrix} (-k-u:\sigma_1,\sigma_2),\left(\frac{1}{2}-k-u-\lambda:\sigma_1,\sigma_2\right),A_3:C_3 \\ B_3,(-k-u+m:\sigma_1,\sigma_2),(-k-u-m-2\lambda:\sigma_1,\sigma_2):D_3\end{matrix}\right\} \quad (4.3)$$

while respecting the conditions (3.1) with $r=2$ and $\sigma_i>0$ $(i=1,2)$.

**Corollary 4**

$$\int_0^1 (1-y^2)^{\lambda-\frac{1}{2}}y^{2u}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2)I^{0,n:V}_{p_i,q_i,\tau_i;R;W}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\(xy^2)^{\sigma_2}\end{matrix}\middle|\begin{matrix}A_3:C_3\\B_3:D_3\end{matrix}\right\}dy$$

$$=\frac{\sqrt{\pi}}{\Gamma(\lambda)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+2\lambda)}{m!}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_p)_k}{(\beta_1)_k\dots(\beta_q)_k}\Omega\left(\alpha_{p+k},\beta_{q+k},0\right)\frac{z^{2k}}{k!}$$

$$\times I^{0,n_1+2;m_2,n_2;m_3,n_3}_{p_i+2,q_i+2;r;p_{i'},q_{i'};r';p_{i''},q_{i''};r''}\left\{\begin{matrix} x^{\sigma_1} \\ x^{\sigma_2}\end{matrix}\middle| \begin{matrix} (-k-u:\sigma_1,\sigma_2),\left(\frac{1}{2}-k-u-\lambda:\sigma_1,\sigma_2\right),A_3:C_3 \\ B_3,(-k-u+m:\sigma_1,\sigma_2),(-k-u-m-2\lambda:\sigma_1,\sigma_2):D_3\end{matrix}\right\} \quad (4.4)$$

while respecting the conditions (3.1) with $r = 2$ and $\sigma_i > 0$ $(i = 1,2)$.

(iii) Converting the multivariable Aleph-function into multivariable H-function [7,8], and using following notations to simplify the formulas

$$A_4 = \left(a_j; \alpha_j', \dots, \alpha_j^{(r)}\right)_{1,p}; \; B_4 = \left[\left(b_j; \beta_j', \dots, \beta_j^{(r)}\right)_{1,q}\right]$$

$$C_4 = \left(c_j', \gamma_j'\right)_{1,p_1}; \dots; \left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,p_r}; \; D_4 = \left(d_j', \delta_j'\right)_{1,q_1}; \dots; \left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,q_r}$$

**Corollary 5**

$$\int_0^1 \boldsymbol{C_m^\lambda}(1-2y^2)\,(1-y^2)^{\lambda-\frac{1}{2}} y^{2\lambda+2u}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2) H_{p,q;W}^{0,n:V}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\(xy^2)^{\sigma_2}\end{matrix}\middle|\begin{matrix}A_4:C_4\\B_4:D_4\end{matrix}\right\} dy$$

$$= \frac{\sqrt{\pi}}{4^\lambda \Gamma(\lambda)} \frac{(-1)^m \Gamma(m+2\lambda)}{m!} \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \Omega\left(\alpha_{p+k}, \beta_{q+k}, 0\right) \frac{z^{2k}}{k!}$$

$$\times H_{p+2,q+2;W}^{0,n+2;V}\left\{\begin{matrix}x^{\sigma_1}\\x^{\sigma_2}\end{matrix}\middle|\begin{matrix}(-k-u:\sigma_1,\dots\sigma_r),\left(\frac{1}{2}-k-u-\lambda:\sigma_1,\dots\sigma_r\right),A_4:C_4\\B_4,(-k-u+m:\sigma_1,\dots\sigma_r),(-k-u-m-2\lambda:\sigma_1,\dots\sigma_r):D_4\end{matrix}\right\}$$

(4.5)

valid under the conditions (3.1) with $\tau_i, \tau_{i'}, \dots, \tau_{i^{(r)}} \to 1$.

**Corollary 6**

$$\int_0^1 (1-y^2)^{\lambda-\frac{1}{2}} y^{2u}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2) H_{p,q;W}^{0,n:V}\left\{\begin{matrix}(xy^2)^{\sigma_1}\\(xy^2)^{\sigma_2}\end{matrix}\middle|\begin{matrix}A_4:C_4\\B_4:D_4\end{matrix}\right\} dy$$

$$= \frac{\sqrt{\pi}}{\Gamma(\lambda)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m+2\lambda)}{m!} \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \Omega\left(\alpha_{p+k}, \beta_{q+k}, 0\right) \frac{z^{2k}}{k!}$$

$$\times H_{p+2,q+2;W}^{0,n+2;V}\left\{\begin{matrix}x^{\sigma_1}\\x^{\sigma_2}\end{matrix}\middle|\begin{matrix}(-k-u:\sigma_1,\dots\sigma_r),\left(\frac{1}{2}-k-u-\lambda:\sigma_1,\dots\sigma_r\right),A_4:C_4\\B_4,(-k-u+m:\sigma_1,\dots\sigma_r),(-k-u-m-2\lambda:\sigma_1,\dots\sigma_r):D_4\end{matrix}\right\}$$

(4.6)

by respecting the conditions (3.2) with $\tau_i, \tau_{i'}, \dots, \tau_{i^{(r)}} \to 1$.

(iv) Converting multivariable Aleph function into single variable Aleph function, and noting

$$A_5 = \left[\left(a_j, A_j\right)_{1,n}\right], \left[\tau_i\left(a_{ji}, A_{ji}\right)_{n+1,p_i}\right]; B_5 = \left[\left(b_j, B_j\right)_{1,m'}\right], \left[\tau_i\left(b_{ji}, B_{ji}\right)_{m'+1,q_i}\right]$$

**Corollary 7**

$$\int_0^1 \boldsymbol{C_m^\lambda}(1-2y^2)\,(1-y^2)^{\lambda-\frac{1}{2}}y^{2\lambda+2u}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2)\aleph\left\{(xy^2)^\sigma\middle|\begin{matrix}A_5\\ B_5\end{matrix}\right\}dy$$

$$=\frac{\sqrt{\pi}}{4^\lambda\Gamma(\lambda)}\frac{(-1)^m\Gamma(m+2\lambda)}{m!}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\Omega\left(\alpha_{p+k},\beta_{q+k},0\right)\frac{z^{2k}}{k!}$$

$$\times$$

$$\aleph_{p_i+2,q_i+2,\tau_i;r}^{m',n+2}\left\{x^\sigma\middle|\begin{matrix}(-k-u:\sigma),\left(\frac{1}{2}-k-u-\lambda:\sigma\right),A_5\\ B_5,(-k-u+m:\sigma),(-k-u-m-2\lambda:\sigma)\end{matrix}\right\} \tag{4.7}$$

while respecting the conditions of Aleph function of one variable [11].

**Corollary 8**

$$\int_0^1 (1-y^2)^{\lambda-\frac{1}{2}}y^{2u}\,{}_pB_q(\alpha_s,\beta_t:(zy)^2)\aleph\left\{(xy^2)^\sigma\middle|\begin{matrix}A_5\\ B_5\end{matrix}\right\}dy$$

$$=\frac{\sqrt{\pi}}{\Gamma(\lambda)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+2\lambda)}{m!}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\Omega\left(\alpha_{p+k},\beta_{q+k},0\right)\frac{z^{2k}}{k!}$$

$$\times\aleph_{p_i+2,q_i+2,\tau_i;r}^{m',n+2}\left\{x^\sigma\middle|\begin{matrix}(-k-u:\sigma),\left(\frac{1}{2}-k-u-\lambda:\sigma\right),A_5\\ B_5,(-k-u+m:\sigma),(-k-u-m-2\lambda:\sigma)\end{matrix}\right\} \tag{4.8}$$

while respecting the conditions of Aleph function of one variable [11].

**Remarks**

Taking $r=r'=r''=1$ so we get similar relationships with the H-function of two variables [17]. We have the similar integrals with others special functions of one and several variables, defined by [18], [19,20], [3,21-23] , [16,24], see also [25-27] and [28,29].

We can have more general formulas if the integrand contains several homogeneous hypergeometric functions and several multivariable Aleph-functions by using the same process.

## 5 CONCLUSION

The importance of our all the results lies in their manifold generality. Firstly, by specializing the parameters as well as the variables of the Aleph-function of several complex variables, we obtain many known and new finite integrals. Secondly, by specializing the parameters of the homogeneous hypergeometric function, we can get a big variety of known and new results. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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