Applied Mathematics and Computational Intelligence
Volume 11, No.1, Dec 2022 [146-163]

A Production Inventory Model with Constant Production Rate, Linear Level Dependent Demand and Linear Holding Cost

Alhamdu Atama Madaki[1*], Babangida Sani[2]

[1]Department of Mathematics/Statistics, Isa Mustapha Agwai I Polytechnic, Lafia, Nigeria.
[2] Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria.

[*] Corresponding author: madakiamatamaalhamdu@gmail.com

Received: 19 January 2022; Accepted: 26 May 2022; Available online (In press): 10 June 2022

ABSTRACT

In this paper, a production inventory model is proposed which considers products with limited life and a little amount of decay. In real life problem, there are many scenarios that happened in production inventory which were not taken into consideration by Shirajul Islam and Sharifuddin [19], who formulated a production inventory model and considered both the holding cost and the production rate to be constant. They assumed that the demand is a linear level dependent. Their paper has been modified and extended by considering the holding cost to be linearly dependent on time and the demand rate during production is assumed to be smaller than the demand rate after production. The proposed production inventory model is formulated using systems of differential equations including initial and boundary conditions and typical integral calculus were also used to analyze the inventory problems. These differential equations were solved to give the best cycle length of the model to minimize the inventory cost. A mathematical theorem and proof are presented to establish the convexity of the cost function. From the numerical examples giving to illustrate the application of the model, a Newton-Raphson method has been used to determine the optimal length of ordering cycle to be 0.54814, optimal cycle time=2.3014 (840days), optimal quantity=32.9675 and total optimal average inventory cost per unit time=18.253 and accompanied by sensitivity analysis to see the effects of the parameter changes.

Keywords: Boundary and Initial Conditions, Linear Level Dependent Demand, Linear Holding Cost, Optimal Solution, Production Inventory.

1 INTRODUCTION

Recently, the attention of manufacturers and managers of production inventories have been drawn to the effects of deterioration of items in the business word since the inventories or goods that are manufactured undergoes decay with time. All products have limited life and market demand, and as a result the inventories continues to deplete and some, if not all deteriorate. This deterioration affects the inventories by reducing the quality and quantity of the goods produced which courses an increase on inventory cost. When an item degenerates to a state that it's no longer valuable or lost original purpose, then it is said that deterioration has occurred. Fashionable goods or items such as tomatoes, mangoes, bananas, etc degenerate easily during the storage period.

2 LITERATURE REVIEW

Managers of industries have developed some models of inventory production to save some real-life situations. This is done by developing or constructing good inventory models to consider the situation at hand depending on the nature of the demand in the market. The demands are not normally static but fluctuates from time to time. Based on the nature of the demand, managers of inventories decide how much items to manufacture and when to manufacture.

Harris [1], developed an inventory model that presents the famous Economic Order Quantity (EOQ) formula for the first time. Whitin [2], considered fashionable goods for decaying items at the end of period of the storage. Ghare and Schrader [3], developed an (EOQ) inventory model with constant rate of deterioration. They pointed out in their research that the consumption of the deteriorating items was closely related to a negative exponential function of time. Covert and Philip [4], introduced an inventory model which considered some parameters of Weibull distribution to represent the distribution of the deterioration. The model was modified and extended by Philip [5], considering up to three-parameter Weibull distribution for deterioration. Shah and Jaiswal [6], developed and discoursed an order level inventory model for deteriorating items for constant rate. Aggarawa [7], studied the model of Shah and Jaiswal [6] by correcting the error in it to calculate the average inventory holding cost. The demand rate and the deterioration rate were constant in all the models, also, the replenishment rate was infinite and there was no shortage allowed in inventory. Dave and Patel [8], considered an inventory model for decaying items with time proportional demand, but the demand was taken to be stock dependent and having linear trend. Deb and Chaudhuri [9], studied a model with finite rate of production and a time proportional deterioration rate, following backlogging. Rafaat [10], further review the work of Deb and Chaudhuri [9] by taken into consideration details information that governed the modeling inventory for deteriorating items. Goswami and Chaudhuri [11] also, further extended the model to include the demand rate, production rate and deterioration rate to be all function of time. Jalan and Chaudhuri [12], developed an order model of inventory for degenerating items with no shortages. Teng *et al* [13], studied a model of degenerating items with shortages and they assumed that the demand fluctuates with time positively. Skouri and Papchristos [14], discussed a continuous review inventory model in which there is opportunity cost due to lost sales and replenishment cost due to the linear dependency on the lot size. Ouyang and Cheng [15], discoursed the inventory model for deteriorating items with exponential declining demand and partial backlogging. Chund and Wee [16], developed an integrated two stages production inventory deterioration model for the buyer and the supplier on the basis of stock dependent selling rate considering important items and in time multiple deliveries. Applying inventory replenishment policy, Cheng and Wang [17], discussed an inventory model for deteriorating items with trapezoidal type demand rate which is a piecewise linear function. In the paper, a class of inventory models was developed with time dependent deterioration rate. Kaliraman *et al* [18], discoursed an inventory model of economic production quantity (EPQ) for degenerating items where the deterioration rate was assumed to follow weilbuill distribution with two parameters. The rate of demand was stock dependent and shortages were not allowed. Shirajul Islam and Sharifuddin [19], formulated an inventory model with constant production rate, linear level dependent demand with buffer stock to minimize inventory cost. In their model, they considered the demand to be the same during and after production with a small amount of constant decay. Ali *et al* [20], developed model of an inventory for delay deteriorating items with price and stock depended on demand, fully backlogged shortage and under inflation. The demand function was assumed to be generally dependent on price and stock and when there was shortage then demand would depend only on price of the product. They considered price of the product to be dependent on different kinds of fixed markup rate and the deterioration was assumed to be non-instantaneous. Shortages were not allowed and fully backlogged. Bashair and Lakdere [21], proposed an EOQ inventory model with backlogging and in the presence of delay deterioration. He argued that the time at which deterioration begins is greater than or equal to the time at which backlogging begins in the basic EOQ model and then the optimal policy was determine by the parameters of basic EOQ model. Swagatika *et al*. [22], contributed in the inventory scenarios of items with instantaneous deterioration. They developed and inventory models for both crisp and fuzzy single commodity with three rates of production where the demand rate was a function of both advertisement and selling price. Dharmendra *et al*. [23], discussed an inventory model for deterioration product for multi-product with partial backlogging to consider carbon emission cost under the influence of inflation. Jamil *et al*. [24], proposed a model of an inventory that considered stock dependent demand allowing few defective items in the model, little amount of decay with constant production rate to find out the total optimum inventory cost, time and ordering cycle.

Motivated by Shirajul Islam and Sharifuddin [19], this paper an inventory model is presented with a linear level dependent demand. The demand during production is assumed to be smaller than the demand after production. There is a small amount of decay during and after production. Our main contribution in this paper is that by considering the holding cost to be linearly dependent on time i.e. $h_1 + h_2 t$ and the demand rate during production is different from the demand rate after production.

3 ASSUMPTIONS

The production rate λ is always constant and greater than the demand rate. The rate of decay μ is constant and small. Since the decay is small it is assumed that there is no deterioration cost as in Shirajul Islam and Sharifuddin [19]. The demand rate during production at any instant t is given by $a + bI(t)$, where a and b are constants and satisfying the condition that $\lambda > a + bI(t)$. The demand rate after production is $c + fI(t)$ and assumed to be greater than demand during production at any instant t where f and c are constants. Production starts with little items in the inventory as a safety stock. The inventory level gets to its highest point at the end of production and after which it reduces to the level of the safety stock due to the effects of market demand and degeneration of the items. There are no shortages.

4 NOTATIONS

$I(t)$ = Stock level at any instance t

I_{1h} = Holding cost for un-decayed inventory from $0\ to\ t_1$

I_{2h} =Holding cost for un-decayed inventory from $t_1\ to\ \mathrm{T}_1$

D_{1h} =Holding cost for deteriorated Inventory from $0\ to\ t_1$

D_{2h} =Holding cost for deteriorated Inventory from $t_1\ to\ \mathrm{T}_1$

Q ,Q_1 are the sock levels at time $t = 0,$ and t=t_1 respectively. Here Q is the safety stock.

dt =Very small portion of instance t
K_o =Set up cost
$h_1 + h_2 t$ =Linear holding cost which is time dependent
$TC = TC(T_I)$ =Total average inventory cost per unit time.
t_I =Time when inventory gets to the maximum level
T_I =Total cycle time
Q_1^* =Optimal order quantity
t_I^* =Optimal time for a maximum inventory
T_I^* =Optimal Order Interval
$TC(T_1)^*$ =Optimal average inventory cost per unit time

5 MODEL FORMULATION

The main objective of any business institution is to maximize profit and minimize cost. As a result, all various decisions have to be taken using suitable models. In a production Inventory environment, the demand pattern and production plant dictate the decisions of how and which model to use. The proposed model may be changed to another depending on the situation. In this model, while $t = 0$, the production λ begins from Q inventory and this continues for the whole production cycle. The inventory continues at the rate of $\lambda - a - bI(t) - \mu I(t)$ at $t = 0 \ to \ t_1$.The demand in market is $a + bI(t)$ and $\mu I(t)$ is the deterioration of $I(t)$ inventory at an instance t. From the above information the differential equation of the situation can be formulated as bellow:

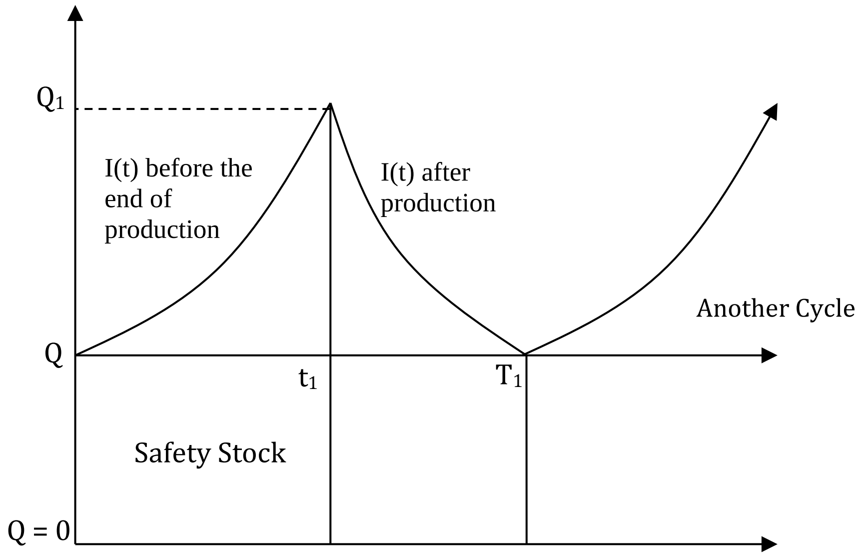

Figure 1: Inventory situation before and after production

$$I\left(t+dt\right)-I\left(t\right)=\left\{\lambda-a-bI\left(t\right)-\mu It\right\}dt$$

$$\lim_{dt\to 0}\frac{I\left(t+dt\right)-I\left(t\right)}{dt}=\lambda-a-bI\left(t\right)-\mu I\left(t\right)$$

$$\frac{d}{dt}I\left(t\right)+\mu I\left(t\right)=\lambda-a-bI\left(t\right)$$

$$\therefore I\left(t\right)=\frac{\lambda-a}{\mu+b}+Ae^{-(\mu+b)t} \tag{1}$$

This is the differential equation that governed the system.

Using initial /matching condition $I\left(t\right)=Q$ at $t=0$ yields

$$\therefore A=Q-\frac{\lambda-a}{\mu+b} \tag{2}$$

$$\therefore I\left(t\right)=\frac{\lambda-a}{\mu+b}+\left(Q-\frac{\lambda-a}{\mu+b}\right)e^{-(\mu+b)t} \tag{3}$$

Using initial/matching condition i.e. at $t=t_1, I\left(t\right)=Q_1$ taking up to the first degree of μ yields

$$Q_1=\frac{\lambda-a}{\mu+b}+\left\{Q-\frac{\lambda-a}{\mu+b}\right\}e^{-(\mu+b)t_1} \tag{4}$$

$$Q_1=\frac{\lambda-a}{\mu+b}+\left\{Q-\frac{\lambda-a}{\mu+b}\right\}\left\{1-\left(\mu+b\right)t_1\right\}$$

$$=Q+\left\{\lambda-a-Q\mu-Qb\right\}t_1 \tag{5}$$

Using equation (3) and considering the total un decayed inventory in the period $t=0$ *to* t_1 and taking the second term of μ yields.

$$I_{1h}=\int_0^{t_1}\left(h_1+h_2t\right)I\left(t\right)dt=\int_0^{t_1}\left(h_1+h_2t\right)\left[\frac{\lambda-a}{\mu+b}+\left\{Q-\frac{\lambda-a}{\mu+b}\right\}e^{-(\mu+b)t}\right]dt$$

$$\therefore I_{1h} = \left[h_1 \left(\frac{\lambda - a}{\mu + b} \right) t + h_1 \left(Q - \frac{\lambda - a}{\mu + b} \right) \frac{e^{-(\mu+b)t}}{-(\mu+b)} + h_2 \left(\frac{\lambda - a}{\mu + b} \right) \frac{t^2}{2} + \left(Q - \frac{\lambda - a}{\mu + b} \right) \left\{ h_2 t \frac{e^{-(\mu+b)t}}{-(\mu+b)} - h_2 \frac{e^{-(\mu+b)t}}{(\mu+b)^2} \right\} \right] \Bigg|_0^{t_1}$$

$$= \frac{h_1(\lambda - a)t_1}{\mu + b} + h_1 \left(Q - \frac{\lambda - a}{\mu + b} \right) \frac{e^{-(\mu+b)t_1} - 1}{-(\mu+b)} + h_2 \left(\frac{\lambda - a}{\mu + b} \right) \frac{t_1^{\,2}}{2} + \left(Q - \frac{\lambda - a}{\mu + b} \right) \left\{ \frac{h_2 t_1 \left(e^{-(\mu+b)t_1} - 1 \right)}{-(\mu+b)} - \frac{h_2 \left(e^{-(\mu+b)t} - 1 \right)}{(\mu+b)^2} \right\}$$

$$= h_1 Q t_1 + \frac{h_1 Q(\mu+b)t_1^{\,2}}{2} - \frac{h_1(\lambda-a)t_1^{\,2}}{2} + \frac{h_2 Q t_1^{\,2}}{2} - \frac{h_2 Q(\mu+b)t_1^{\,3}}{2} + \frac{Q h_2 t_1}{\mu+b}$$
$$+ \frac{h_2(\lambda-a)t_1^{\,3}}{2} - \frac{h_2(\lambda-a)t_1}{(\mu+b)^2} \qquad (6)$$

Now to calculate the holding cost for deteriorated items as follows:

$$D_{1h} = \int_0^{t_1} \mu(h_1 + h_2 t) I(t)\, dt = \mu \int_0^{t_1} (h_1 + h_2 t) \left[\frac{\lambda - a}{\mu + b} + \left\{ Q - \frac{\lambda - a}{\mu + b} \right\} e^{-(\mu+b)t} \right] dt$$

$$D_{1h} = h_1 \mu Q t_1 + \frac{h_1 \mu Q(\mu+b)t_1^{\,2}}{2} - \frac{h_1 \mu(\lambda-a)t_1^{\,2}}{2} + \frac{h_2 \mu Q t_1^{\,2}}{2} - \frac{h_2 \mu Q(\mu+b)t_1^{\,3}}{2}$$

$$+ \frac{h_2 \mu Q t_1}{\mu+b} + \frac{h_2 \mu(\lambda-a)t_1^{\,3}}{2} - \frac{h_2 \mu(\lambda-a)t_1}{(\mu+b)^2} \qquad (7)$$

Also, the inventory changes or reduces on the other side at the rate of $c + fI(t) + \mu I(t)$ at $t = t_1$ to T_1 as production stop after time t_1. The demand after production is assumed to be greater than the demand during production. The inventory reduces to the level of safety stock due to effects of degeneration and the market demands of the items. The same procedure is applied also.

$$I(t + dt) = It + \{-c - fI(t)\} dt - \mu I(t) dt$$

$$I(t + dt - It) = \{-c - fI(t) - \mu I(t)\} dt$$

$$\lim_{dt \to 0} \frac{(t + dt - It)}{dt} = \{-c - fI(t) - \mu I(t)\}$$

$$I(t) = \frac{-c}{\mu + f} + Be^{-(\mu+f)t} \qquad (8)$$

Which is the differential equation that governed the system.

Using initial/matching condition when $t=T_1, I(t)=Q$ yields

$$I(t)=\frac{-c}{\mu+f}+\left(Q+\frac{c}{\mu+f}\right)e^{(\mu+f)(T_1-t)} \tag{9}$$

Using initial /matching condition $I(t)=Q_1$ When $t=t_1$, considering the first term of μ to obtain the equations bellow.

$$Q_1=\frac{-c}{\mu+f}+\left(Q+\frac{c}{\mu+f}\right)e^{(\mu+f)(T_1-t_1)}$$

$$=Q+\left\{c+Q(\mu+f)\right\}(T_1-t_1) \tag{10}$$

Now using Equation (9) to get the holding cost for undecayed inventory during $t=t_1$ to T_1 as

$$I_{2h}=\int_{t_1}^{T_1}(h_1+h_2t)I(t)\,dt=\int_{t_1}^{T_1}(h_1+h_2t)\left[\frac{-c}{\mu+f}+\left\{Q+\frac{c}{\mu+f}\right\}e^{(\mu+f)(T_1-t)}\right]dt$$

$$=\left[\begin{array}{l}h_1\left(\frac{-c}{\mu+f}\right)t+h_1\left(Q+\frac{c}{\mu+f}\right)\left\{\frac{e^{(\mu+f)(T_1-t)}}{-(\mu+f)}\right\}+h_2\left(\frac{-c}{\mu+f}\right)\frac{t^2}{2}+\left(Q+\frac{c}{\mu+f}\right)\\ \left\{\frac{h_2te^{(\mu+f)(T_1-t)}}{-(\mu+f)}-\frac{e^{(\mu+f)(T_1-t)}h_2}{(\mu+f)^2}\right\}\end{array}\right]_{t_1}^{T_1}$$

$$=h_1\left(\frac{-c}{\mu+f}\right)(T_1-t_1)+h_1\left(Q+\frac{c}{\mu+f}\right)\left\{\frac{e^{(\mu+f)(T_1-T_1)}-e^{(\mu+f)(T_1-t_1)}}{-(\mu+f)}\right\}$$
$$+h_2\left(\frac{-c}{\mu+f}\right)\frac{\left(T_1^2-t_1^2\right)}{2}+\left(Q+\frac{c}{\mu+f}\right)\left[h_2(T_1-t_1)\left\{\frac{e^{(\mu+f)(T_1-T_1)}-e^{(\mu+f)(T_1-t_1)}}{-(\mu+f)}\right\}-\left\{\frac{e^{(\mu+f)(T_1-T_1)}-e^{(\mu+f)(T_1-t_1)}}{(\mu+f)^2}\right\}h_2\right]$$

$$\therefore I_{2h}=h_1\left(\frac{-c}{\mu+f}\right)(T_1-t_1)+h_1Q(T_1-t_1)+h_1\left(\frac{c}{\mu+f}\right)(T_1-t_1)+h_2\left(\frac{-c}{\mu+f}\right)\left(\frac{T_1^2-t_1^2}{2}\right)$$

$$+h_2Q\left(T_1^2-2T_1t_1+t_1^2\right)+\frac{h_2Q(T_1-t_1)}{\mu+f}+h_2\left(\frac{c}{\mu+f}\right)\left(T_1^2-2T_1t_1+t_1^2\right)+\frac{h_2c(T_1-t_1)}{(\mu+f)^2} \tag{11}$$

Multiply equation (11) by μ above to get the holding cost for deteriorated items during the period t_1 to T_1 as below

$$D_{2h}=\int_{t_1}^{T_1}\mu\left(h_1+h_2t\right)I\left(t\right)dt=\mu\int_{t_1}^{T_1}\left(h_1+h_2t\right)\left[\frac{-c}{\mu+f}+\left\{Q+\frac{c}{\mu+f}\right\}e^{(\mu+f)(T_1-t)}\right]dt$$

$$=h_1\mu\left(\frac{-c}{\mu+f}\right)\left(T_1-t_1\right)+h_1\mu Q\left(T_1-t_1\right)+h_1\mu\left(\frac{c}{\mu+f}\right)\left(T_1-t_1\right)$$

$$+h_2\mu\left(\frac{-c}{\mu+f}\right)\left(\frac{T_1^2-t_1^2}{2}\right)+h_2\mu Q\left(T_1^2-2T_1t_1+t_1^2\right)+\frac{h_2\mu Q\left(T_1-t_1\right)}{\mu+f}$$

$$+h_2\mu\left(\frac{c}{\mu+f}\right)\left(T_1^2-2T_1t_1+t_1^2\right)+\frac{h_2\mu c\left(T_1-t_1\right)}{\left(\mu+f\right)^2} \tag{12}$$

We equate equations (5) and (10) to get the following equations:

$$Q+\left\{\lambda-a-Q\mu-Qb\right\}t_1=Q+\left\{c+Q\left(\mu+f\right)\right\}\left(T_1-t_1\right)$$

$$\therefore t_1=\frac{\left\{c+Q\left(\mu+f\right)\right\}T_1}{c-a+Q\left(-b+f\right)+\lambda} \tag{13}$$

Now let

$$V=\frac{c+Q\left(\mu+f\right)}{c-a+Q\left(-b+f\right)+\lambda} \tag{14}$$

$$\therefore t_1=VT_1 \tag{15}$$

The total average cost per unit time is given as

$$TC\left(T_1\right)=\frac{K_o+I_{1h}+D_{1h}+I_{2h}+D_{2h}}{T_1} \tag{16}$$

By substituting equations (6), (7), (11), (12), and (15) in equation (16) yields

$$TC(T_1)=\frac{1}{T_1}\left[\begin{array}{l}
K_o+h_1Qt_1-\frac{h_1Q(\mu+b)t_1^{\,2}}{2}+\frac{h_1(\lambda-a)t_1^{\,2}}{2}+\frac{h_2Qt_1^{\,2}}{2}\\
-\frac{h_2Q(\mu+b)t_1^{\,3}}{2}+\frac{h_2Qt_1}{\mu+b}+\frac{h_2(\lambda-a)t_1^{\,3}}{2}-\frac{h_2(\lambda-a)t_1}{(\mu+b)^2}\\
+h_1\mu Qt_1-\frac{h_1\mu Q(\mu+b)t_1^{\,2}}{2}+\frac{h_1\mu(\lambda-a)t_1^{\,2}}{2}+\frac{h_2\mu Qt_1^{\,2}}{2}\\
-\frac{h_2\mu Q(\mu+b)t_1^{\,3}}{2}+\frac{h_2\mu Qt_1}{\mu+b}+\frac{h_2\mu(\lambda-a)t_1^{\,3}}{2}-\frac{h_2\mu(\lambda-a)t_1}{(\mu+b)^2}\\
+h_1\left(\frac{-c}{\mu+f}\right)(T_1-t_1)+h_1Q(T_1-t_1)+h_1\left(\frac{c}{\mu+f}\right)(T_1-t_1)+h_2\left(\frac{-c}{\mu+f}\right)\left(\frac{T_1^2-t_1^{\,2}}{2}\right)\\
+h_2Q\left(T_1^2-2T_1t_1+t_1^{\,2}\right)+\frac{h_2Q(T_1-t_1)}{\mu+f}+h_2\left(\frac{c}{\mu+f}\right)\left(T_1^2-2T_1t_1+t_1^{\,2}\right)+\frac{h_2c(T_1-t_1)}{(\mu+f)^2}\\
+h_1u\left(\frac{-c}{\mu+f}\right)(T_1-t_1)+h_1\mu Q(T_1-t_1)+h_1u\left(\frac{c}{\mu+f}\right)(T_1-t_1)+h_2u\left(\frac{-c}{\mu+f}\right)\left(\frac{T_1^2-t_1^{\,2}}{2}\right)\\
+h_2\mu Q\left(T_1^2-2T_1t_1+t_1^{\,2}\right)+\frac{h_2\mu Q(T_1-t_1)}{\mu+f}+h_2\mu\left(\frac{c}{\mu+f}\right)\left(T_1^2-2T_1t_1+t_1^{\,2}\right)+\frac{h_2\mu C(T_1-t_1)}{(\mu+f)^2}
\end{array}\right]$$

$$\begin{aligned}
TC(T_1)=&\frac{K_o}{T_1}+\frac{h_1Q(1+\mu)t_1}{T_1}-\frac{h_1Q(\mu+b)(1+\mu)t_1^{\,2}}{2T_1}+\frac{h_2(\lambda-a)(1+\mu)t_1^{\,2}}{2T_1}+\frac{h_2Q(1+\mu)t_1^{\,2}}{2T_1}\\
&-\frac{h_2Q(\mu+b)(1+\mu)t_1^{\,3}}{2T_1}+\frac{h_2Q(1+\mu)t_1}{(\mu+b)T_1}+\frac{h_2(\lambda-a)(1+\mu)t_1^{\,3}}{2T_1}-\frac{h_2(\lambda-a)(1+\mu)t_1}{(\mu+b)^2T_1}\\
&+\frac{h_1Q(1+\mu)(T_1-t_1)}{T_1}-\frac{h_2\left(\frac{c}{\mu+f}\right)(1+\mu)\left(T_1^2-t_1^{\,2}\right)}{2T_1}+\frac{h_2Q(1+\mu)(T_1-t_1)}{(\mu+f)T_1}\\
&+\frac{h_2Q(1+\mu)\left(T_1^2-2T_1t_1+t_1^{\,2}\right)}{T_1}+\frac{h_2\left(\frac{c}{\mu+f}\right)(1+\mu)\left(T_1^2-2T_1t_1+t_1^{\,2}\right)}{T_1}+\frac{h_2c(1+\mu)(T_1-t_1)}{(\mu+f)^2T_1}
\end{aligned}$$

By substituting $t_1=VT_1$ so that the last equation becomes

$$TC(T_1)=\frac{K_o}{T_1}+h_1Q(1+\mu)V-\frac{h_1(\mu+b)(1+\mu)V^2T_1}{2}+\frac{h_2(\lambda-a)(1+\mu)V^2T_1}{2}+$$
$$\frac{h_2Q(1+\mu)V^2T_1}{2}-\frac{h_2Q(\mu+b)(1+\mu)V^3T_1^2}{2}+\frac{h_2Q(1+\mu)V}{\mu+b}+$$
$$\frac{h_2(\lambda-a)(1+\mu)V^3T_1^2}{2}-\frac{h_2(\lambda-a)(1+\mu)V}{(\mu+b)^2}+h_1Q(1+\mu)(1-V)$$
$$-\frac{h_2\left(\frac{c}{\mu+f}\right)(1+\mu)(1-V^2)T_1}{2}+h_2Q(1+\mu)(1-2V+V^2)T_1+\frac{h_2c(1+\mu)(1-V)}{(\mu+f)^2}$$
$$+h_2\left(\frac{c}{\mu+f}\right)(1+\mu)(1-2V+V^2)T_1+\frac{h_2Q(1+\mu)(1-V)}{(\mu+f)} \qquad (17)$$

The main objective is to find the value of T_1 which gives the minimum variable cost per unit time. The necessary and sufficient condition to minimize $TC(T_1)$ are respectively:

$$\frac{dTC(T_1)}{dT_1}=0 \text{ and } \frac{d^2TC(T_1)}{dT_1^2}>0$$

Now, differentiate equation 17 with respect to T_1 as follows:

$$\frac{dTC(T_1)}{dT_1}=-\frac{K_o}{T_1^2}-\frac{h_1(\mu+b)(1+\mu)V^2}{2}+\frac{h_2(\lambda-a)(1+\mu)V^2}{2}$$
$$+\frac{h_2Q(1+\mu)V^2}{2}-h_2Q(\mu+b)(1+\mu)V^3T_1+h_2(\lambda-a)(1+\mu)V^3T_1$$
$$-\frac{h_2\left(\frac{c}{\mu+f}\right)(1+\mu)(1-V^2)}{2}+h_2Q(1+\mu)(1-2V+V^2)$$
$$+h_2\left(\frac{c}{\mu+f}\right)(1+\mu)(1-2V+V^2) \qquad (18)$$

This is now equated to zero so as to obtain the T_1 which reduces the cost function.

Theorem 5.1: If $Q(\mu+b)<(\lambda-a)$ then the cost function is convex.

Proof: From equation (18), we take the second derivative as follows:

$$\frac{d^2TC(T_1)}{dT_1^2}=\frac{2K_o}{T_1^3}-h_2Q(\mu+b)(1+\mu)V^3+h_2(\lambda-a)(1+\mu)V^3 \tag{19}$$

Therefore, $\frac{d^2TC(T_1)}{dT_1^2}>0$ provided $h_2Q(\mu+b)(1+\mu)V^3<h_2(\lambda-a)(1+\mu)V^3$

$$\therefore Q(\mu+b)<(\lambda-a)$$

Therefore, equation (17) shows that the cost function is convex in T_1, then there is optimality inT_1 provided $Q(\mu+b)<(\lambda-a)$ is satisfied.

6 MODEL DEMONSTRATION

A numerical illustration is provided to demonstrate the developed model. The values of various parameters are as follows: K_o= ₦100 Set up cost, λ = 50, Q = 10, h_1 = 3, h_2 = 2, b = 0.4, f = 0.8, μ = 0.01, a = 4 and c = 5. Note that the values of the parameters satisfy theorem 1. Now we substitute the above values of parameters into equations (18) and (19) to compute for T_1 using Newton-Raphson method the solution T_1^* obtained from equations (18) and (19) is now put into equations (5), (15) and (17) to obtain the optimal solution as Q_1^* = 32.9675, t_1^* = 0.54814, $TC(T_1)^*$ = ₦18.45253 and T_1^* = 2.3014(840days).

7 EFFECTS OF THE PARAMETER ON THE MODEL

We carefully examine the effects of each parameter K_o, λ, Q, h_1, h_2, b, f, μ, a and c on the optimal length of ordering cycle t_l^*, optimal cycle time T_1^*, optimal quantity Q_1^* and the total average inventory cost $TC(T_1)^*$. The sensitivity analysis is carried out by changing each of the parameters by 50%, 25%, 10%, 5%, -5%, -10%, -25%, -50% taking one parameter at a time and leaving other parameters unchanged.

Table 1: The effects of the parameter changes on the model demonstration 1 to see some changes on the variables of T_1^*, t_1^*, Q_1^* and $TC(T_1)^*$

Parameter	**% Change in Parameter**	**T_1^***	**t_l^***	**Q_1^***	**$TC(T_1)^*$**
K_o	50%	2.7808(1016 days)	0.66234	37.7521	38.1275
	25%	2.5534 (933 days)	0.60820	35.4832	28.75036
	10%	2.4082 (880 days)	0.57361	34.0343	22.70157
	5%	2.3562 (861 days)	0.56122	33.5143	20.6011
	0%	**2.3014 (840 days)**	**0.54814**	**32.9675**	**18.45253**
	-5%	2.24915 (821 days)	0.535746	32.44776	16.25375
	-10%	2.19182 (801 days)	0.522695	31.8742	14.0011
	-25%	2.011369 (735 days)	0.479626	30.0695	6.859817
	-50%	1.66032 (607 days)	0.3954	26.5693	-6.75602
λ	50%	2.2219(812days)	0.36388	34.34105	13.80372
	25%	2.260274(825days)	0.438661	33.86314	15.55688
	10%	2.28491(835days)	0.49894	33.39720	17.099
	5%	2.2932(838days)	0.52245	33.1964	17.73786
	0%	**2.3014 (840 days)**	**0.54814**	**32.9675**	**18.45253**
	-5%	2.3123(845days)	0.577605	32.7334	19.2632
	-10%	2.3205(848days)	0.608701	32.4353	20.18536
	-25%	2.3452(857days)	0.7229	31.2536	23.87677
	-50%	2.3644(864days)	1.03243	27.4485	36.16761
Q	50%	1.99726(729days)	0.60093	38.947006	25.1871
	25%	2.1315(779days)	0.57572	36.0321	21.59405
	10%	2.22704(815days)	0.55934	34.2043	19.66405
	5%	2.2658(827days)	0.55433	33.6134	19.05193
	0%	**2.3014 (840 days)**	**0.54814**	**32.9675**	**18.45253**
	-5%	2.3425(856days)	0.54272	32.3491	17.86424
	-10%	2.3863(872days)	0.53713	31.7263	17.28472
	-25%	2.5314(925days)	0.51925	29.7864	15.5741
	-50%	2.86301(1046days)	0.48873	26.48640	12.6101

Parameter	**% Change in Parameter**	**T_1^***	**t_l^***	**Q_1^***	**$TC(T_1)^*$**
h_1	50%	2.30411(841days)	0.548797	32.99459	33.56201
	25%	2.30411(841days)	0.548797	32.99459	26.00731
	10%	2.30411(841days)	0.548797	32.99459	21.47449
	5%	2.30411(841days)	0.54813	32.9674	19.96355
	0%	**2.3014 (840 days)**	**0.54814**	**32.9675**	**18.45253**
	-5%	2.3014(840days)	0.54814	32.9674	16.9421
	-10%	2.3014(840days)	0.54814	32.9674	15.4307
	-25%	2.3014(840days)	0.54814	32.9674	10.89791
	-50%	2.3014(840days)	0.54814	32.9674	3.343207
h_2	50%	1.9041(696days)	0.45352	29.0035	-11.223
	25%	2.073973(757days)	0.493983	30.6984	4.079186
	10%	2.2027(804days)	0.524700	31.9836	12.83317
	5%	2.24930(823days)	0.53579	32.4481	15.66662
	0%	**2.3014(840days)**	**0.54814**	**32.9675**	**18.45253**
	-5%	2.358904(862days)	0.561848	33.5413	21.18754
	-10%	2.41924(885days)	0.57624	34.1430	23.86777
	-25%	2.63293(962days)	0.62715	36.2761	31.52568
	-50%	3.17532(1160days)	0.75633	41.6891	42.55763
a	50%	2.309589(844days)	0.57086	32.77734	19.09261
	25%	2.306849(843days)	0.55962	32.88864	18.76458
	10%	2.30411(841days)	0.552818	32.9423	18.5755
	5%	2.3014(840days)	0.55084	32.9580	18.51371
	0%	**2.3014 (840 days)**	**0.54814**	**32.9675**	**18.45253**
	-5%	2.3014(840days)	0.54625	32.99327	18.39197
	-10%	2.3014(840days)	0.54423	33.0191	18.33201
	-25%	2.3014(840days)	0.53841	33.0953	18.15552
	-50%	2.29589(838days)	0.52772	33.1642	17.87253
b	50%	2.30411(841days)	0.56954	32.7238	84.30588
	25%	2.30411(841days)	0.55896	32.8614	61.05371
	10%	2.30411(841days)	0.552818	32.9420	38.94667
	5%	2.30411(841days)	0.55081	32.9415	29.42021
	0%	**2.3014 (840 days)**	**0.54814**	**32.9675**	**18.45253**
	-5%	2.3014(840days)	0.54622	32.99327	5.740142
	-10%	2.3014(840days)	0.54421	33.0191	-9.1059
	-25%	2.3014(840days)	0.53835	33.09548	-72.3034
	-50%	2.3014(840days)	0.52951	33.24721	-323.236

Parameter	% Change in Parameter	T_1^*	t_l^*	Q_1^*	$TC(T_1)^*$
c	50%	2.243836(819days)	0.60883	35.50741	7.42149
	25%	2.2712(830days)	0.57940	34.2785	12.84602
	10%	2.290411(836days)	0.56135	33.5175	16.18791
	5%	2.2959(839days)	0.55481	33.2441	17.31666
	0%	**2.3014 (840 days)**	**0.54814**	**32.9675**	**18.45253**
	-5%	2.309589(843days)	0.54211	32.7132	19.59593
	-10%	2.3151(847days)	0.535227	32.4265	20.7472
	-25%	2.33798(853days)	0.515224	31.58788	24.24294
	-50%	2.3726(876days)	0.47987	30.07235	30.2165
F	50%	2.2438(820days)	0.65031	37.24923	-9.3467
	25%	2.2849(834days)	0.60530	35.36234	2.699325
	10%	2.2986(840days)	0.57262	33.9923	11.54658
	5%	2.3014(840days)	0.56081	33.4985	14.87838
	0%	**2.3014 (840 days)**	**0.54814**	**32.9675**	**18.45253**
	-5%	2.3014(840days)	0.53538	32.42978	22.30974
	-10%	2.2986(840days)	0.521623	31.5570	26.50142
	-25%	2.27952(833days)	0.47795	30.00284	41.85644
	-50%	2.17532(794days)	0.38864	26.26346	87.4984
μ	50%	2.29593(838days)	0.54892	32.9730	20.78036
	25%	2.2986(839days)	0.54859	32.96935	19.62411
	10%	2.3014(840days)	0.54863	32.9793	18.92301
	5%	2.3014(840days)	0.54845	32.97328	18.68807
	0%	**2.3014 (840 days)**	**0.54814**	**32.9675**	**18.45253**
	-5%	2.30411(841days)	0.54860	32.98856	18.21642
	-10%	2.30411(841days)	0.54848	32.98253	17.97961
	-25%	2.3041(841days)	0.547701	32.96417	17.26549
	-50%	2.3066(843days)	0.547400	32.9610	16.06238

8 DISCUSSION OF RESULTS

From the results obtained in Table 1, it can be deduced as follows:

The effects of the set up cost, K_0, on the variables T_1*, t_1*, Q_1*, and $TC(T_1)$* is that all increase. This implies that increase in set up cost will result in the increase of the optimal time for maximum inventory t_1*, optimal cycle time T_1*, optimal production quantity Q_1* and total average inventory cost per unit time $TC(T_1)$*. This is clearly expected since excess stocking is encouraged as a result of high set up cost. The total average inventory cost per unit time $TC(T_1)$* is therefore expected to increase due to increase in stocking cost. The variable T_1*, t_1* and Q_1* all increase due to high set up cost as well as stock holding cost.

When there is a change in the value of the production rate λ, the variables T_1*, t_1* and $TC(T_1)$* reduces while Q_1* increases. This is expected because high production rate leads to shorter cycle time T_1* especially if the demand rate after production is more than that during production. This will in turn reduce $TC(T_1)$***.** Q_1* increases since production rate increases.

When the value of the safety stock Q increases, the variables T_1* reduces while the t_1*, Q_1*, and $TC(T_1)$* increase. This is because inventory produced takes shorter time to finish hence the optimal cycle T_1*reduces. On the other hand, the optimal time for maximum inventory t_1* and optimal quantity Q_1* increase probably because Q is much. The total average inventory cost is increased due to increase in the holding cost for the safety stock.

The effects of the constant part of the holding cost h_1, the variables T_1*, t_1* and Q_1* remain unchanged while $TC(T_1)$* increases. This is because as the demand increases, the optimal average cost $TC(T_1)$* increases. On the other hand, the parameter h_1 does not affect optimal time for maximum inventory t_1* and optimal quantity Q_1* based on equations (13) and (5). They are not very sensitive to h_1.

The stock depended part of the holding cost h_2 increases, the variables T_1*, t_1*, Q_1*, and $TC(T_1)$* all reduces. This is expected since if the stock dependent part of the holding cost is higher, the model will force a reduction in the value of the optimal stock Q_1*. Therefore, T_1*, t_1* and Q_1*will all reduce and this will in turn cause $TC(T_1)$* to reduce.

The parameter, a, of the constant part of the demand rate during production increases or changes, while the variables T_1*, t_1* and $TC(T_1)$* increase while the value of Q_1* reduces. This is expected since if a is higher, the demand rate is higher and this will increase the optimal cycle time T_1*, the time for maximum inventory t_1* as well as the average total cost per unit time $TC(T_1)$*. Q_1* reduces probably due to increase in t_1*.

When there is change in the value of stock dependent part of the demand during production, the variables T_1*almost remains unchanged. t_1* and $TC(T_1)$* increase while the value of Q_1* reduces. Increasing the value of the parameter b, increases the demand and this will in turn increase both T_1* and the total average inventory cost per unit time. The model will then force a reduction of the optimal production quantity Q_1*, to reduce stock holding cost.

When there is a change in the value of the parameter c of the constant part of the demand after production, the decision variables T_1* and $TC(T_1)$* reduces while the values of t_1*and Q_1* increase. This is expected since if c increases the demand rate increases so Q_1* and t_1* increase. The high stock will take less time to finish due to high demand and the total average inventory cost per unit time will reduce.

The value of the parameter d, of the stock dependent demand rate after production changes, the variables t_1*and Q_1*increase, while the values of TC(T_1)* reduces. This is expected since if d is higher, the demand rate is higher, and this will increase the optimal cycle time T_1* though in our case T_1* is unstable. The time for maximum inventory t_1* as well as the optimal quantity Q_1* also increase due to higher demand. Thus the model will seek to lower value of total average inventory cost per unit time TC(T_1)*.

The effects of the change of deterioration rate μ, on the decision variables is that T_1* reduces while TC(T_1)* and t_1*increase but Q_1* is unstable. This is because deterioration forces the model to lower the value of T_1*. Also due to deterioration, t_1* will increase so as to make up for what is going to deteriorate. As for TC(T_1)*, it increases due to increase in deterioration cost.

9 CONCLUSION REMARKS

This paper presents a mathematical model of inventory production with constant production rate and linear level dependent demand. The demand during production is assumed to be different from the demand after production even though they are both linear level dependent. There is little amount of constant decay during and after production. A mathematical theorem and proof are presented to show the convexity of the cost function. Also, Newton-Raphson method has been used to determine the optimal solutions of the developed cost minimization model and a numerical illustration is given to demonstrate the application of the developed model. The main objective of the proposed model is to get the optimal length of ordering cycle, optimal cycle time, optimal quantity and total optimal average of the inventory cost per unit time. This paper concludes with notations, assumptions, development of the model, numerical examples and sensitivity analysis.

REFERENCES

[1] F. W. Haris, “Operations and Costs,” in *A.W. Shaw Company*, Chicago, pp. 48-50, 1957.

[2] T. M. Whitin, “Theory of Inventory Management,” Princeton University Press, Princeton, NJ, pp. 62-72, 1957.

[3] P. M. Ghare and G. F. Schrade, “A Model for an Exponential Decaying Inventory,” *Journal of Industrial Engineering*, vol.14, pp. 238-243, 1963.

[4] R. P. Covert and G. C. Philip, “An EOQ Model with Weibull Distribution Deterioration,” *AIIE Transactions*, vol. 5, pp. 323-326, 1973.

[5] G. C. Philip, “A Generalized EOQ Model for Items with Weibull Distribution Deterioration,” *AIIE Transactions*, vol. 6, pp. 159-162, 1974.

[6] Y. K. Shah and M. C. Jaiswal, “An order-Level Inventory Model for Asylum with Constant rate of Deterioration,” *Opsearch*, vol. 14, pp. 174-184, 1977.

[7] S. P. Aggarwal, "A Note on an order Inventory Model for a System with Constant rate of Deterioration," *Opsearch*, vol. 15, pp. 184-187, 1978.

[8] U. Dave and L. K. Patel, "Policy Inventory Model for Deteriorating Items with Time Proportional Demand," *Journal of the Operations Research Society*, vol. 32, pp. 137-142, 1981.

[9] M. Deb and K. S. Chaudhuri, "An EOQ Model for Items with Finite rate of Production and Variable rate of Deterioration," *Opsearch*, vol. 23, pp. 175-181, 1986.

[10] F. Rafaat, "Survey of Literature On Continuously Deteriorating Inventory Model," *Journal of the Operational Research Society*, vol. 42, pp. 27-37, 1991.

[11] A. Goswami and K. S. Chaudhuri, "Variation of order-level Inventory Models for Deteriorating items," *International Journal of Production Economics*, vol. 27, pp. 1105-1110, 1992.

[12] A. K. Jalan and K. S. Chaudhuri, "Structural Properties of an Inventory System with Deteriorating and trended demand," *International Journal of System Science*, vol. 30, pp. 627-633, 1999.

[13] J. T. Teng, M. S. Chern, and H. L. Yang, "Deterministic Lot Size Inventory Models with Shortages and Deterioration for Fluctuating Demand," *Operations Research Letters*, vol. 24, pp. 65-72, 1999.

[14] K. Skouri and S. Papachristos, "A Continuous Review Inventory Model, with Deteriorating Items, Time varying Demand, Linear Replenishment Cost and Partially Time Varying Backlogging," *Applied Mathematical Modeling*, vol. 26, pp. 603-617, 2002.

[15] N Ouyang, and X. Cheng, "An Inventory Model for Deteriorating Items with Exponential Declining Demand and Partial Backlogging," *Yugoslav Journal of Operations Research, vol.* 15, pp. 277-288,2005.

[16] C. J. Chund and H. E. Wee, "Scheduling and Replenishment Plan for an Integrated Deteriorating Inventory Model with Stock-Dependent Selling Rate," *An International Journal of Advanced Manufacturing Technology*, vol. 35, pp. 665-679, 2008.

[17] M. B. Cheng and G. Q. Wang, "A Note on the Inventory Model for Deteriorating Items with Trapezoidal Type Demand Rate," *Computers and Industrial Engineering*, vol. 56, pp. 1296-1300, 2009.

[18] N. K. Kaliraman, S. Raji, and H. Chaudhry, "An EPQ Inventory Model for Deteriorating Items with weibull Deterioration Under Stock Dependent Demand," *International Journal of Scientific and Technology Research*, vol. 4, no. 01, pp. 232-236, 2015.

[19] I. U. Shirajul and M. Sharifuddin, "A production inventory model of constant production rate and demand of level dependent linear trend," *American Journal of Operations Research*, vol. 6, pp. 61-70, 2016.

[20] A. S. Ali, H. M. Abu, U. Sharif, and K. Al-Amin, "Non-Instantaneous Deterioration Inventory Model with Price and Stock Dependent Demand for Fully Backlogged Shortages Under Inflation," *International Journal of Business Forecasting and Marketing Intelligence*, vol. 3, no. 2, pp. 152-164, 2017.

[21] A. Bashair and B. Lakdere, "Economic Order- Type Models for Non- Instantaneous Deterioration items and Backlogging," *RAIRO- Oper. Res*, vol. 52, no. 3, pp. 895-901, 2018

[22] S. Swagatika, A. Milu, and M. N. Mitali, "A Three Rates of EOQ/EPQ Model for Instantaneous Deteriorating items involving Fuzzy Parameter Under Shortages," *International Journal of Innovative Technology and Engineering*, vol. 8, no. 8, pp. 405-418, 2019.

[23] Y. Dharmendra, S. R. Singh, and M. Sarin, "Inventory Model Considering Deterioration, Stock-Dependent and Ram-Type Demand with Reserve Money and Carbon Emission," *International Journal of Recent Technology and Engineering*, vol. 8. No. 5, pp. 2277-3878, 2020.

[24] M. U. Jamil, I. K. Shirajul, R. K. Aminura, and M. S. Uddin, "An inventory model of Production with Level Dependent Demand Alowing Few Defective items," *America Journal of Operations Research*, vol. 11, pp. 1-14, 2021.

[25] R. B. Mishra, "Optimum Production Lot-Size Model for a System with Deteriorating Inventory," *International Journal of Production Research*, vol. 13, pp. 495-505, 1975.