# A STUDY OF LIE GROUP METHOD AND ITS APPLICATION TO SOLVE THE UNSTEADY TRANSONIC FLOW

by

**MOHAMMAD HUSKHAZRIN BIN KHARUDDIN**
**(1332120846)**

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**Institute of Engineering Mathematics**
**UNIVERSITI MALAYSIA PERLIS**

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# LIST OF SYMBOLS

| | |
|---|---|
| $X$ | Banach Space |
| $f$ | Function |
| $\mathbb{R}$ | Set of Real numbers |
| $\subseteq$ | Subset |
| $\in$ | Element |
| $C_K$ | Continuous Transformation Group |
| $\partial$ | Partial derivative |

**Kajian Tentang Kaedah Kumpulan Lie dan Aplikasinya Untuk Menyelesaikan Aliran Transonik Tak Mantap**

## ABSTRAK

Di dalam disertasi ini, persamaan pergerakan tak linear menerangkan tentang aliran transonik tak mantap dalam koordinat cartesian. Kaedah yang dikenali sebagai kumpulan Lie yang menurunkan persamaan pembezaan separa tak linear kepada persamaan pembezaan biasa berdasarkan struktur simetri pendasar telah digunakan. Kaedah Lie ini amat berguna dalam menurunkan suatu persamaan yang kompleks kepada persamaan pembezaan biasa yang ringkas. Dengan menggunakan teori Lie, kumpulan penuh satu parameter penjelmaan sangat halus yang memberikan hasil kepada persamaan pergerakan tak beza diterbitkan sepenuhnya melalui Aljabar Lie. Subkumpulan bagi kumpulan penuh digunakan selepas itu untuk mengurangkan jumlah bilangan pembolehubah bergerak balas di dalam sistem ini. Pengurangan ini diteruskan sehingga persamaan terbitan biasa dicapai. Penyelesaian tepat jenis siri yang dihasilkan daripada penyelesaian persamaan terbitan yang dikurangkan akan didapati dimana juga merupakan penyelesaian sebenar jenis siri bagi persamaan aliran transonik. Disebabkan, persamaan yang menggambarkan gerakan adalah sangat kompleks dan tak linear, keadah kumpulan Lie menjadi pilihan yang sesuai untuk menangani persamaan tak linear.

**The Study of Lie Group Method and Its Application to Solve the Unsteady Transonic Flow**


## ABSTRACT

The non-linear equations of motion describing the unsteady transonic flow in cartesian coordinates are considered in this dissertation. A method known as Lie group which reduce the non-linear partial differential equation to an ordinary differential equation on the basis of the underlying symmetry structure has been used. The Lie method is quite useful in reducing a complex equation to an easy-to-handle ordinary differential equation. By employing the Lie theory, the full one-parameter infinitesimal transformation group leaving the equations of motion invariance is derived along with its associated Lie algebra. Subgroups of the full group are then used to obtain a reduction by one in the number of independent variables in the system. These reductions are continued until an ordinary differential equation is reached. A series type exact solution of these reduced ordinary differential equation is obtained which leads to a series type exact solution of the unsteady transonic flow equation. The Lie group method seems to be an appropriate choice to handle these nonlinear equation.

# CHAPTER 1

# INTRODUCTION

## 1.1 Introduction

This chapter describes the general matter for this research. It consisting of background of the problem, statement of the problem, objectives of the study, scope of the study, output expected from the study along with the significance of the study. Lastly, lay out of the dissertation is presented.

## 1.2 Background of Research

In the nineteenth century, Sophus Lie novel study on the integration theory of differential equations has played a significant role in investigations into different mathematical aspects of soliton systems governed by continuous equations during the past few decades.

Due to the applicability of Lie group method to handle, fairly large number of non-linear problems, it has been widely used by researcher from different areas of research. (Ames & Nucci, 1985; Biswas & Krishnan, 2011; Clarkson & Kruskal, 1989; Fakhar, & Kara, 2012; Fakhar, Kara, Morris & Hayat, 2013; Hongyan, 2009; Levi, Menyuk, & Winternitz, 1994; Ludlow, Clarkson, & Bassom, 1999; Masemola, Fakhar & Kara, 2013; Nucci & Clarkson, 1992; Shoufeng, 2007; Singh & Gupta, 2005; Xiaoda, Chunli, Zhang, & Yishen, 2004)

The main objective of the research is to present the theory related to Lie group method and after that to use Lie group method to solve a relatively simple equation as an application of Lie group method.

## 1.3 Problem Statement

How to handle the non-linearity in an unsteady transonic equation to come up with new exact mathematical expressions describing the motion of the equation?

## 1.4 Research Objectives

The aims to do the research are:

a) To study the basic concepts of Lie group theory.
b) To reduce the non-linear unsteady transonic flow equation to equivalent transonic flow equation where the variables less by one using Lie group method.
c) To reduce the equivalent transonic flow equation to equivalent ordinary differential equation using Lie group method.
d) To find the exact solution of non-linear unsteady transonic flow equation via the equivalent ordinary differential equation.

## 1.5 Research Scope

In this dissertation some fundamental concepts of Lie group method is introduced. By using Lie group, the non-linear unsteady transonic flow equation is solved. The infinitesimal transformations and operators related to the non-linear unsteady transonic flow equation are discussed. Next, with help of the infinitesimal transformation group, the generators and the Lie algebra of the non-linear unsteady transonic flow equation are obtained. Finally, the reduction is done by using symmetries and thereby finding the exact solution.

## 1.6 Significance of the Study

The group theory is a combination of algebra and analysis that is worth to investigate. The Lie group algebra obtained by Lie group theory not only reflects the internal structure of the equation but also helpful in reducing the order of the differential equations. The reduced ordinary differential equation is more easy to handle as compared to the original one. It is worth to mention that the Lie group procedure able to produce exact solution of highly non-linear equation, which could not be tackled by the other methods.



## 1.7 Dissertation Organisation

In this dissertation, Chapter 1 covers the general material regarding this research. Chapter 2 describes the importance definitions that are useful for this research. In Chapter 3, the infinitesimal transformation group for the problem under consideration is obtained by using classical method. The result obtained in this chapter is needed in the next chapter. Chapter 4 is the vital part in this dissertation. It discusses about the generator of the group and the corresponding Lie Algebra table. In this chapter, the solution of non-linear unsteady transonic flow equation is obtained. At the end of the chapter, the validation of the result is presented. The last chapter is Chapter 5, which contains conclusion of the dissertation and the recommendations for further research.

# CHAPTER 2

# LITERATURE REVIEW

## 2.1 Introduction

In this chapter, some preliminaries are presented, basic definitions and few examples for understanding of definitions and which ultimately helpful for understanding the Lie group theory. The basic definitions include: Finite Transformation Group; Lie Group, and Lie Algebra.

## 2.2 Finite Transformation Group

Finite transformation groups or one-parameter transformation group is a fundamental concept in Lie group theory defined as follows, (Bluman & Anco, 2002; Ibragimov, 1999; Olver, 1986)



**Definition 2.1** (Finite Transformation Groups)

Let *X* be a Banach space and there is a mapping $f : X \times \mathbb{R} \to X$ ; the set $\{ f(a) : a \in \mathbb{R} \}$ denoted by $f(\mathbb{R})$ is called Finite Transformation Group of *X* or one-parameter transformation group of the space *X*, if the following properties are satisfied.

a) $f_0(x) = x$ for any $x \in X$

b) $f_b(f_a(x)) = f_{a+b}(x)$ for any $x \in X$ and $a, b \in \mathbb{R}$

c) There is an open interval $I \subseteq \mathbb{R}$ symmetric with respect to zero such that $a \in I$ and $f_a(x) = x$ for any $x$ implies $a = 0$.

Continuous Transformation Group, denoted as $C_K$ is a group $f(\mathbb{R})$ such that $C_K \in (X \times \mathbb{R})$. It means that the points $f_a(x)$ will display a continuous curve since parameter $a$ is varied continuously. The clear picture of the definition for Continuous Transformation Group and Finite Transformation Group are shown in the following two examples.



**Example 2.1** (Translation Group)

Suppose that $x_0 \in X$, $x_0 \neq 0$, and define $f : X \times \mathbb{R} \to X$ by

$$f(x;a) = x + a\,x_0 . \tag{2.2.1}$$

***Solution***

i) $f(x;0) = x + (0)x_0$,

$$= x.$$

ii) Suppose

$$f(x;a) = x + a\,x_0$$

Then,

$$f\left(f\left(x;a\right),b\right)=\left(x+ax_0\right)+bx_0$$

$$f\left(f\left(x;a\right),b\right)=x+\left(a+b\right)x_0$$

$$f\left(f\left(x;a\right),b\right)=f\left(x,a+b\right)$$

From (i) and (ii), its show that the transformation shifts a vector $x$ by following the direction $x_0$, it's safe to say that $f$ generates a continuous transformation group. Hence the path curves of the group are straight lines parallel to the vector in $x_0$.



**Example 2.2** (Rotation Group)

Suppose that $\left(x,y\right)\in X$ where $X=\mathbb{R}^2$, and define $f:X\times\mathbb{R}\rightarrow X$ by

$$f\left(x,y;a\right)=\left(x\cos a-y\sin a,x\sin a+y\cos a\right). \tag{2.2.2}$$

***Solution***

i) $f\left(x,y;0\right)=\left(x\cos 0-y\sin 0,x\sin 0+y\cos 0\right),$

$$=\left(x(1)-y(0),x(0)+y(1)\right),$$

$$=\left(x,y\right).$$

ii) Suppose

$$f\left(x,y;a\right)=\left(x\cos a-y\sin a,x\sin a+y\cos a\right).$$

Then,

$$f\left(f\left(x,y;\mathrm{a}\right);b\right)=\begin{bmatrix}\left(x\cos a-y\sin a\right)\cos b-\left(x\sin a-y\cos a\right)\sin b,\\ \left(x\cos a-y\sin a\right)\sin b+\left(x\sin a+y\cos a\right)\cos b\end{bmatrix},$$

$$f\left(f\left(x,y;\mathrm{a}\right);b\right)=\begin{bmatrix}x\cos a\cos b-y\sin a\cos b-x\sin a\sin b-y\cos a\sin b,\\ x\cos a\sin b-y\sin a\sin b+x\sin a\cos b+y\cos a\cos b\end{bmatrix},$$

$$f\left(f\left(x,y;\mathrm{a}\right);b\right)=\begin{bmatrix}x\left(\cos a\cos b-\sin a\sin b\right)-y\left(\sin a\cos b+\cos a\sin b\right),\\ x\left(\cos a\sin b+\sin a\cos b\right)-y\left(\sin a\sin b-\cos a\cos b\right)\end{bmatrix},$$

$$f\left(f\left(x,y;\mathrm{a}\right);b\right)=\begin{bmatrix}x\left(\cos a\cos b-\sin a\sin b\right)-y\left(\sin a\cos b+\cos a\sin b\right),\\ x\left(\cos a\sin b+\sin a\cos b\right)+y\left(-\sin a\sin b+\cos a\cos b\right)\end{bmatrix},$$

since, $-y\left(\sin a\sin b-\cos a\cos b\right)=y\left(-\sin a\sin b+\cos a\cos b\right)$.

Then,

$$f\left(f\left(x,y;\mathrm{a}\right);b\right)=\begin{bmatrix}x\cos\left(a+b\right)-y\sin\left(a+b\right),\\ x\sin\left(a+b\right)+y\cos\left(a+b\right)\end{bmatrix}=f\left(x,y;a+b\right).$$

iii) To satisfy property (c), let choose $I=\left(-\pi,\pi\right)$.

Therefore, rotation group is a Finite Transformation Group.

## 2.3 Infinitesimal Transformation

Follow from the definition of one-parameter transformation group, the definition of Infinitesimal Transformation is defined as follows (Ibragimov, 1999);

**Definition 2.2** (Infinitesimal Transformation)

Suppose a one-parameter transformation group (in general),

$$f^i(x;a) \qquad i = 1, 2, ..., n, \tag{2.3.1}$$

and expand it into the Taylor Series with parameter *a* in a neighbourhood of $a = 0$.

Then by using the initial condition

$$f^i(x;0) \qquad i = 1, 2, ..., n, \tag{2.3.2}$$

Infinitesimal Transformation is defined as

$$f^i(x;\mathrm{a}) = \bar{x}^l \approx x^i + a\xi^i(x) \qquad i = 1, 2, ..., n, \tag{2.3.3}$$

where

$$\xi^i(x) = \frac{\partial}{\partial a}\left[f_a^i(x)\right]_{a=0}. \tag{2.3.4}$$

Then, the Infinitesimal Operator can be defined as

$$X = \xi^i(x)\frac{\partial}{\partial x^i}. \tag{2.3.5}$$

The following example explains the definition in detail.

**Example 2.3**

In the case of rotations, suppose

$$\bar{x} = x\cos a + y\sin a, \qquad \bar{y} = y\cos a - x\sin a$$

Note:

$$f^1(x,a) = \bar{x}; \qquad f^2(x,a) = \bar{y},$$

therefore,

$$\begin{aligned}
\xi^1(x) &= \frac{\partial}{\partial a}\left[f^1(x,a)\right]\Big|_{a=0} \\
&= \frac{\partial}{\partial a}[\bar{x}]\Big|_{a=0} \\
&= \frac{\partial}{\partial a}[x\cos a + y\sin a]\Big|_{a=0} \\
&= -x\sin a + y\cos a\big|_{a=0} \\
&= -x\sin(0) + y\cos(0) \\
&= -x(0) + y(1) \\
&= y
\end{aligned}$$

$$\begin{aligned}
\xi^2(x) &= \frac{\partial}{\partial a}\left[f^2(x,a)\right]\Big|_{a=0} \\
&= \frac{\partial}{\partial a}[\bar{y}]\Big|_{a=0} \\
&= \frac{\partial}{\partial a}[y\cos a - x\sin a]\Big|_{a=0} \\
&= -y\sin a - x\cos a\big|_{a=0} \\
&= -y\sin(0) - \mathrm{x}\cos(0) \\
&= -y(0) - x(1) \\
&= -x,
\end{aligned}$$

thus the Infinitesimal Transformation is

$\bar{x} \approx x + ya$ and $\bar{y} \approx y - xa$, with the Infinitesimal Operator

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \ .$$

From the infinitesimal transformation, one can find the associated finite transformation group. (Bluman & Anco, 2002)

## 2.4 Invariance

Invariance is defined on a group transformation *G*, as a function *F*(*x*) if *F* remains unaltered when one moves along any path curve of the group *G*. (Bluman & Anco, 2002)

**Definition 2.3** (Invariance)

Suppose L(*u*) = 0 have a solution of the partial differential equation such that $u = \theta(x, y)$ with boundary conditions.

$$B_i(x, y, u) = 0, \qquad i = 1, 2, ..., k \tag{2.4.1}$$

$$\Gamma_i(x, y) = 0, \qquad i = 1, 2, ..., k \tag{2.4.2}$$

The transformation group leaves the above invariance if and only if $u'(x, \mathrm{y}, \theta(x, y); \varepsilon)$ satisfy the above in prime coordinates.

For example, let $G$ be the group of Galilean transformations $\bar{x} = x + ay$ and $\bar{y} = y$ with generator $X = y\frac{\partial}{\partial x}$, then the invariant of this group is $u = y$.

## 2.5 Lie Group

Below is the definition of Lie Group. (Bluman & Anco, 2002)

**Definition 2.4** (Lie Group)

A real Lie Group is defined as a group of a finite-dimensional real smooth manifold and also group operations of multiplication and inversion are smooth maps. Smoothness of the multiplication of a group is defined as a mapping

$$\mu : G \times G \to G \text{ and } \mu(x, y) = xy \qquad (2.5.1)$$

where $\mu$ denotes as a smooth mapping of the product manifold $G \times G$ into $G$. Hence two requirements as stated can be combined to the single requirement such as the mapping

$$(x, y) \mapsto x^{-1}y,$$

that shows it is a smooth mapping of the product manifold into $G$.

## 2.6 Lie Algebra

Finite or one-parameter Transformation Group can be obtained by using infinitesimal generator *X*. To reduce the number of independent variables in the given differential equation, Lie Algebra is used. The theory of Lie Algebras is now well-developed fields in modern mathematics.

According to Ovsiannikov (1964), if given an infinitesimal transformation group that leave partial differential equation invariance, then it can be associated with Lie Algebra (Olver, 1986). The following is the definition of Lie Algebra (Ibragimov, 1999).

**Definition 2.5** (Lie Algebra)

A Lie Algebra is defined as a vector space *L* of operators $X = \xi_1^i(x)\frac{\partial}{\partial x^i}$ , if the following properties are satisfied:

a) *the mapping is closed*

$$[X_1, X_2] \in L \quad \text{for any } X_1, X_2 \in L \ . \tag{2.6.1}$$

b) *the mapping is bilinear*

$$[aX_1 + \mathrm{b}X_2, X_3] = a[X_1, X_3] + b[X_2, X_3],$$

$$[X_1, \mathrm{a}X_2 + bX_3] = a[X_1, X_2] + b[X_1, X_3] \quad \text{for any } a, b \in F; \quad X_1, X_2, X_3 \in L \ . \tag{2.6.2}$$

c) *the mapping is antisymmetric*

$$[X_1, X_2] + [X_2, X_1] = 0 \qquad \text{for any } X_1, X_2 \in L \ . \tag{2.6.3}$$

d) *satisfying Jacobi's Identity*

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0 \quad \text{for any } X_1, X_2, X_3 \in L \ . \tag{2.6.4}$$

## 2.7 Non-linear Unsteady Transonic Flow Equation

The method presented in this dissertation is illustrated in the next chapter by considering a form of the non-linear unsteady transonic flow equation. The equation that going to be solved is the equation of non-linear unsteady transonic flow,

$$u_t + u_{xx} - u_y u_{yy} = 0 \tag{2.7.1}$$

where $t$ is time variable, $x$ and $y$ are the space variables and $u$ is the velocity component in $x$-direction. The infinitesimal transformation group for this equation and its associated Lie Algebra will be derived. Then, an invariant solution is obtained by using these groups. The solution for the steady transonic flow equation by using the methods presented in this dissertation have been done by Boisvert (1982).

## 2.8 Conclusion

This chapter finished with the basic important definition, along with the brief description of non-linear unsteady transonic flow equation.