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Mass Transport Velocity in a Two-dimensional Wave Tank

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Abstract: *Waves occurred on the surface of water in two-dimensional wave tank are studied. Deriving vorticity equation from incompressible Navier-Stokes equation, the boundary condition on the free surface, water bottom and flap surface are discussed. Using separation of variable method, velocity potential takes the general form in terms of eigen function. To obtain the values of horizontal and vertical velocity components, the boundary conditions on the water bottom and the flap surface are used and then mass transport velocity has been derived. Finally, time average of horizontal velocity component at mean water level is generalized.*

Keywords: *Mass Transport Velocity, Navier-Stokes Equation, Stream Function.*

1. Introduction

A basic theory for the mass transport velocity in water waves in viscous fluid and of finite depth has been formulated. Longuet-Higgins [1, 2, 3] presented a simple physical model to obtain the Lagrangian characters including particle motion, mass transport, the Lagrangian wave period and the Lagrangian mean level for the surface waves that cannot directly obtain throughout the entire flow field. Hsien- Kuo et al. [4] discussed gravity waves in water of uniform depth of governing equation in Lagrangian form. If a small neutrally buoyant float is placed in a wave tank and its trajectory was traced as waves pass by, a small mean motion in the direction of the waves can be observed. There are two approaches for examining the mass transport, one is the Eulerian velocity which involves a fixed point to measure the mean flux of mass and other is the Lagrangian velocity which involves moving with water particles. The stream function for mass transport is calculated from the products of the first order eigen functions for progressive and local waves. Naciri and Mei

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[5] also constructed Lagrangian asymptotic solutions of the non-linear water waves. Iskandarani and Liu [6] discussed on mass transport in two-dimensional wave tank. Many Lagrangian asymptotic solutions of the non-linear water waves have been developed such as Buldakov et al [7], Clamond [8] and Constantin [9]. Boufermel et al. [10] formulated velocity of mass transport taking as variable to model acoustic streaming. Frode [11] studied mass transport velocity in shallow water waves reflected at right angles from an infinite and straight coast in a rotating ocean. In this paper, mass transport velocity has been formulated using boundary conditions at water bottom and the flap surface. Finally, we see that the time average of horizontal velocity component is zero at mean water level which is same as Dean and Dalrymple [12].

2. Mathematical Formulation

2.1. Governing Equation

The motion of the water surface consists in general regular waves for local disturbance progressing on a constant water depth $(z = h)$ as shown in Fig.1.

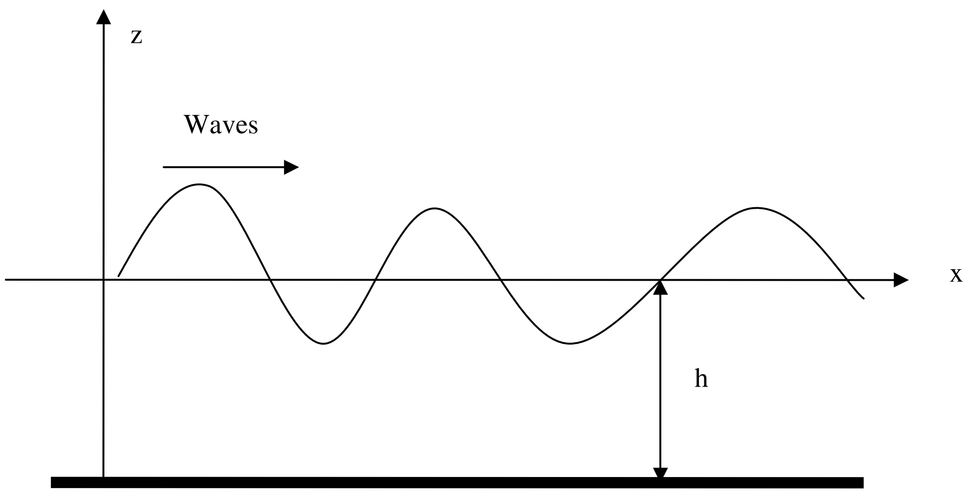

Figure1: Wave Train

Mass transport velocity ($\varepsilon^2\overline{U_1}, \varepsilon^2\overline{W_1}$), defined as the average of the Lagrangian velocity which is in terms of particle velocity and stream function can be written as

$$\overline{U}_1 = \overline{u}_1 + \overline{\int \frac{\partial \psi_1}{\partial z} dt \frac{\partial^2 \psi_1}{\partial x \partial z}} - \overline{\int \frac{\partial \psi_1}{\partial x} dt \frac{\partial^2 \psi_1}{\partial z^2}} = \frac{\partial \Psi}{\partial z}. \tag{1}$$

and

$$\overline{W_1} = \overline{w}_1 - \overline{\int \frac{\partial \psi_1}{\partial z} dt \frac{\partial^2 \psi_1}{\partial x^2}} + \overline{\int \frac{\partial \psi_1}{\partial x} dt \frac{\partial^2 \psi_1}{\partial x \partial z}} = -\frac{\partial \Psi}{\partial x}. \tag{2}$$

For the mass transport velocity $\overline{U}$, the stream function Ψ can be written as

$$\Psi = \overline{\psi}_2 + \overline{\int \frac{\partial \psi_1}{\partial z} dt \frac{\partial \psi_1}{\partial x}}. \tag{3}$$

The Navier-Stokes equation for incompressible flow is

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)\overline{u} = -\frac{1}{\rho}\nabla p + \overline{F} + \upsilon \nabla^2 \overline{u}. \tag{4}$$

Where u *and* w denote the horizontal and vertical velocity components respectively, p is the pressure gradient, $\overline{F}$ is the external body force and υ denotes the kinematic coefficient of viscosity where, $\upsilon = \frac{\mu}{\rho}$.

Taking curl on both sides of Eq. (4), we have

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)\overline{\xi} = \upsilon \nabla^2 \overline{\xi} \text{ , since, vorticity } \overline{\xi} = \nabla \times \overline{u} \tag{5}$$

$$\left[\because\ \overline{F} = -\rho \overline{g} = -\rho(\nabla \phi),\ and\ \nabla \times \nabla\left[-\left\{\frac{p}{\rho} + \phi\right\}\right] = 0\right]$$

Now, the vorticity equation from the above equation is in the following

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z} - \upsilon \nabla^2\right)\nabla^2 \psi = 0. \tag{6}$$

where, $\xi = \nabla \times \overline{u} = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \psi}{\partial z}\right)$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\psi = \nabla^2 \psi.$$

Now,

$$\nabla^2 \psi_1 = \upsilon \int \nabla^4 \psi_1 dt. \tag{7}$$

[taking first-order term of Eq. (6)]

and

$$\overline{\left(u_1 \frac{\partial}{\partial x} + w_1 \frac{\partial}{\partial z}\right)\nabla^2 \psi_1} = \upsilon \nabla^4 \overline{\psi_2}. \tag{8}$$

[taking time average of second-order term of Eq. (6)]

Using Eq. (7) in Eq. (8), we have

$$\nabla^4 \overline{\psi_2} = \overline{\left(u_1 \frac{\partial}{\partial x} + w_1 \frac{\partial}{\partial z} \right) \int \nabla^4 \psi_1 dt}. \tag{9}$$

From Eq. (3), the stream function for the mass transport velocity $\overline{U}$ can be written as

$$\nabla^4 \Psi = \overline{\left(u_1 \frac{\partial}{\partial x} + w_1 \frac{\partial}{\partial z} \right) \int \nabla^4 \psi_1 dt} + \nabla^4 \overline{\int \frac{\partial \psi_1}{\partial z} dt \frac{\partial \psi_1}{\partial x}}. \tag{10}$$

Assuming ψ_1 satisfies Laplace equation, Eq. (10) which is considered as the boundary layer theory can be rewritten as

$$\nabla^4 \Psi = \nabla^4 \overline{\int \frac{\partial \psi_1}{\partial z} dt \frac{\partial \psi_1}{\partial x}}. \tag{11}$$

According to Longuet-Higgins [1], boundary condition for Ψ on the bottom of water is given by

$$\left(\frac{\partial \Psi}{\partial n} \right)_{n=\infty} = \frac{5-3i}{4i\omega} q_{s1}^{(\infty)} \frac{\partial}{\partial s} q_{s1}^{(\infty)*}. \tag{12}$$

where $q_{s1}^{(\infty)}$ represents the first order tangential velocity at outer edge of the boundary layer at the fixed wall and superscript * represents the complex conjugate of the value and the boundary condition at free surface is

$$\left(\frac{\partial^2 \Psi}{\partial n^2} \right)_{n=\infty} = \frac{4}{i\omega} \frac{\partial}{\partial s} q_{n1}^{(0)} \frac{\partial}{\partial s} q_{s1}^{(0)*}. \tag{13}$$

2.2. Linear Water Wave Theory

Linear water waves are of small amplitude for which we can linearise the equations of motion. The water wave motion is represented by a velocity potential $\Phi(x, z, t)$ which is considered as

$$\Phi(x, z, t) = \mathrm{Re}\{\phi(x, z) e^{i\omega t}\}. \tag{14}$$

We assume that the bottom surface is of constant depth at $z = -h$. The water surface is at $z = 0$ and the region of interest is $-h < z < 0$.

Now the equations are in the following

$$\phi_{xx} + \phi_{zz} = 0, -h < z < 0. \tag{15}$$

$$\left. \begin{aligned} \frac{\partial \phi}{\partial z} &= 0, \ z = -h, \\ \frac{\partial \phi}{\partial z} &= K\phi, \ z = 0. \end{aligned} \right\} \tag{16}$$

The last expression can be obtained from combining of the following two equations

$$\left.\begin{aligned} \frac{\partial \phi}{\partial z} &= \pm i\omega\eta,\ z=0, \\ \mp i\omega\phi &= g\eta,\ z=0. \end{aligned}\right\} \tag{17}$$

where $K = \dfrac{\omega^2}{g} = k \tanh kh$ (18)

$$= -k_n \tan k_n h, \quad n = 1,2,.............$$

Eq. (15) is solved by the separation of variable method. So the velocity potential ϕ are taken as

$$\phi(x,z) = U(x)P(z). \tag{19}$$

Substituting this value in Eq. (15), we have

$$\frac{1}{U}\frac{d^2U}{dx^2} = -\frac{1}{P}\frac{d^2P}{dz^2}. \tag{20}$$

The L.H.S of Eq. (20) is only a function of x and the R.H.S of Eq. (20) is only a function of z. So each term is constant. Therefore,

$$\frac{1}{U}\frac{d^2U}{dx^2} = -\frac{1}{P}\frac{d^2P}{dz^2} = \pm k^2. \text{ (say)} \tag{21}$$

Taking positive sign, we have from Eq. (21),

$$\frac{d^2P}{dz^2} + k^2 P = 0. \tag{22}$$

and

$$\frac{d^2U}{dz^2} - k^2 U = 0. \tag{23}$$

Solving Eq. (22), we obtain

$P(z) = Ae^{ikz} + Be^{-ikz}$, where A and B are arbitrary constants. Therefore, Eq. (19) becomes

$$\phi(x,z) = \left(Ae^{ikz} + Be^{-ikz}\right)U(x). \tag{24}$$

Applying boundary condition $\dfrac{\partial \phi}{\partial z}\Big|_{z=h} = 0$ in Eq. (24), we get $A = Be^{-2ikh}$.

Hence,

$P(z) = D\cos k(z-h).$

$D = 2Be^{-ikh}$ = constant.

Therefore, Eq. (24) becomes

$$\phi(x,z) = D\cos k\,(z-h)U(x). \tag{25}$$

Again, solving Eq. (23), we have

$U = A_1 e^{kx} + B_1 e^{-kx}$, where A_1 *and* B_1 are arbitrary constants.

If we set $A_1 = 0$ *and* $B_1 = 1$, then $U = e^{-kx}$.

Therefore, Eq. (25) becomes

$$\phi(x,z) = D\cos k\,(z-h)e^{-kx}. \tag{26}$$

Using Eq. (17) in Eq. (26), we have

$$\frac{\partial\phi}{\partial z} = -Dk\sin khe^{-kx} = -i\omega\eta,\ z=0$$

$$\Rightarrow Dk\sin khe^{-kx} = i\omega\eta,\ z=0. \tag{27}$$

This expression for ϕ *and* η become more convenient if we write $\eta = ae^{-kx}$, Eq. (27) can be written as

$$D = \frac{ai\omega}{k\sin kh}.$$

Substituting the value of D in Eqs. (24) and (26), we have

$$\phi(x,z) = \frac{ai\omega}{k\sin kh}\cos k\,(z-h)e^{-kx}. \tag{28}$$

and

$$P(z) = \frac{ai\omega}{k\sin kh}\cos k(z-h). \tag{29}$$

The boundary condition at the free surface gives $\omega^2 = -gk\tan kh$, which is the dispersion relation for a free surface.

The above equation is not really the dispersion relation for a free surface, it would be better to refer to it as a transcendental equation. If we solve for all roots in the complex plane, we find that the first root is a pair of imaginary roots. We denote the imaginary solutions of this equation by $k_0 = \pm ik$ and real solution by $k_n, n \geq 1$. The k of the imaginary solution is the wave number. So we put $\omega^2 = gk\tanh kh$.

Finally, we define the function $P(z)$ as

$$P(z) = \frac{a_n i\omega}{k_n\sin k_n h}\cos k_n(z-h),\ n \geq 0.$$

as the vertical eigen function in the open water region.

Also, we can write $U = e^{-k_n x}$.

Hence, the velocity potential eigen function expansion can be written as

$$\phi(x,z) = \frac{a_n i\omega}{k_n \sin k_n h} \cos k_n (z-h) e^{-k_n x}. \tag{30}$$

Using Eqs. (18) in (30), we have

$$\phi(x,z) = -\frac{a_n i\omega}{K \cos k_n h} \cos k_n (z-h) e^{-k_n x}. \tag{31}$$

Similarly, taking negative sign, we have from Eq. (20),

$$\frac{d^2 P}{dz^2} - k^2 P = 0. \tag{32}$$

and

$$\frac{d^2 U}{dz^2} + k^2 U = 0. \tag{33}$$

Solving Eq. (32), we have

$P(z) = A_2 e^{kz} + B_2 e^{-kz}$, where A_2 *and* B_2 are arbitrary constants. Therefore, Eq. (18) becomes

$$\phi(x,z) = \left(A_2 e^{kz} + B_2 e^{-kz}\right) U(x). \tag{34}$$

Applying boundary condition $\frac{\partial \phi}{\partial z}|_{z=h} = 0$ in Eq. (34), we get $A_2 = B_2 e^{-2kh}$.

Hence,

$P = D_1 \cosh k(z-h)$, where, $D_1 = 2B_2 e^{-kh}$ = constant.

Therefore, Eq. (34) becomes

$$\phi(x,z) = D_1 \cosh k (z-h) U(x). \tag{35}$$

Again, solving Eq. (33), we have

$U = A_3 e^{ikx} + B_3 e^{-ikx}$, where A_3 *and* B_3 are arbitrary constants.

If we set $A_3 = 0$ *and* $B_3 = 1$, then $U = e^{-ikx}$.

Therefore, Eq. (35) becomes

$$\phi(x,z) = D_1 \cosh k (z-h) e^{-ikx}. \tag{36}$$

Using Eq. (17) in Eq. (36), we have

$$\frac{\partial \phi}{\partial z} = D_1 k \sinh khe^{-ikx} = i\omega\eta,\ z = 0.$$

$$\Rightarrow D_1 k \sinh khe^{-ikx} = i\omega\eta,\ z = 0.$$

This expression for ϕ *and* η become more convenient if we write $\eta = ia_0 e^{-ikx}$, we get

$$D_1 = -\frac{a_0\omega}{k \sinh kh}.$$

Hence, from Eq. (36), we have

$$\phi(x,z) = -\frac{a_0\omega}{k \sinh kh} \cosh k\,(z-h) e^{-ikx}. \tag{37}$$

Using Eq. (18) in Eq. (37), we have

$$\phi(x,z) = -\frac{a_0\omega}{K \cosh kh} \cosh k\,(z-h) e^{-ikx}.$$

Therefore,

$$\phi(x,z) = -\frac{a_0\omega}{K \cosh kh} \cosh k\,(z-h) e^{-ikx} - \sum_{n=1}^{\infty} \frac{a_n i\omega}{K \cos k_n h} \cos k_n\,(z-h) e^{-k_n x}.$$

[Using eigen function expansion]

In general form, the velocity potential can be expressed as

$$\Phi(x,z,t) = \mathrm{Re}\left[-\frac{a_0\omega}{K \cosh kh} \cosh k\,(z-h) e^{-i(kx-\omega t)} - \sum_{n=1}^{\infty} \frac{a_n i\omega}{K \cos k_n h} \cos k_n\,(z-h) e^{-k_n x} e^{i\omega t} \right]$$

$$\Rightarrow \Phi(x,z,t) = \mathrm{Re}\left[-\frac{a_0\omega}{k \cosh kh} \cosh k\,(z-h) e^{-i(kx-\omega t)} + \sum_{n=1}^{\infty} \frac{a_n i\omega}{k_n \cos k_n h} \cos k_n\,(z-h) e^{-k_n x} e^{i\omega t} \right]$$

[$\because$ from Eq. (18), $K = k = -k_n$]

$$\Phi(x,z,t) = \mathrm{Re}\left[i^2 \frac{a_0\omega}{k \cosh kh} \cosh k\,(z-h) e^{-i(kx-\omega t)} + \sum_{n=1}^{\infty} \frac{a_n i\omega}{k_n \cos k_n h} \cos k_n\,(z-h) e^{-k_n x} e^{i\omega t} \right]$$

$$\Rightarrow \Phi(x,z,t) = \mathrm{Re}\left[i^2 \frac{\theta_0 a_0\omega K}{k \cosh kh} \cosh k\,(z-h) e^{-i(kx-\omega t)} + \sum_{n=1}^{\infty} \frac{\theta_0 a_n i\omega K}{k_n \cos k_n h} \cos k_n\,(z-h) e^{-k_n x} e^{i\omega t} \right].$$

[Assuming $\theta_0 = \dfrac{1}{K}$]

$$\therefore \Phi(x,z,t) = \mathrm{Re}\left\{ i\omega\theta_0 \phi_1(x,z) e^{i\omega t} \right\}. \tag{38}$$

where,

$$\phi_1(x,z) = \frac{Ka_0 i}{k\cosh kh}\cosh k(z-h) e^{-ikx} + \sum_{n=1}^{\infty} \frac{Ka_n}{k_n \cos k_n h}\cos k_n (z-h) e^{-k_n x}. \tag{39}$$

2.3. Formulation of mass transport velocity

The first order potential $\phi_1(x,z)$ can be written by the eigen function expansion,

$$\phi_1(x,z) = ia_0 \frac{K}{k}\frac{\cosh k(z-h)}{\cosh kh} e^{-ikx} + \sum_{n=1}^{\infty} a_n \frac{K}{k_n}\frac{\cos k_n(z-h)}{\cos k_n h} e^{-k_n x}, \text{ [From Eq. (39)]}$$

$$= ia_0 \frac{K}{k}\frac{\cosh k(z-h)}{\cosh kh}(\cos kx - i\sin kx) + \sum_{n=1}^{\infty} a_n \frac{K}{k_n}\frac{\cos k_n(z-h)}{\cos k_n h} e^{-k_n x}. \tag{40}$$

Substituting this value in Eq. (38), we get

$$\Phi(x,z,t) = \mathrm{Re}\left[i\omega\theta_0 \left\{ \begin{array}{l} ia_0 \dfrac{K}{k}\dfrac{\cosh k(z-h)}{\cosh kh}(\cos kx - i\sin kx) \\ + \displaystyle\sum_{n=1}^{\infty} a_n \frac{K}{k_n}\frac{\cos k_n(z-h)}{\cos k_n h} e^{-k_n x} \end{array} \right\} e^{i\omega t} \right]$$

$$= \mathrm{Re}\left[\omega\theta_0 \left\{ \begin{array}{l} -a_0 \dfrac{K}{k}\dfrac{\cosh k(z-h)}{\cosh kh}(\cos kx - i\sin kx) \\ + i\displaystyle\sum_{n=1}^{\infty} a_n \frac{K}{k_n}\frac{\cos k_n(z-h)}{\cos k_n h} e^{-k_n x} \end{array} \right\} (\cos\omega t + i\sin\omega t) \right]$$

$$= \mathrm{Re}\left[\omega\theta_0 \left\{ \begin{array}{l} -a_0 \dfrac{K}{k}\dfrac{\cosh k(z-h)}{\cosh kh}\{(\cos kx\cos\omega t + \sin kx \sin\omega t) + i(\cos kx \sin\omega t - \sin kx\cos\omega t)\} \\ + i\displaystyle\sum_{n=1}^{\infty} a_n \frac{K}{k_n}\frac{\cos k_n(z-h)}{\cos k_n h} e^{-k_n x}\cos\omega t - \sum_{n=1}^{\infty} a_n \frac{K}{k_n}\frac{\cos k_n(z-h)}{\cos k_n h} e^{-k_n x}\sin\omega t \end{array} \right\} \right]$$

$$= \mathrm{Re}\left[\omega\theta_0 \left\{ \begin{array}{l} -a_0 \dfrac{K}{k}\dfrac{\cosh k(z-h)}{\cosh kh}\cos(kx-\omega t) - \displaystyle\sum_{n=1}^{\infty} a_n \frac{K}{k_n}\frac{\cos k_n(z-h)}{\cos k_n h} e^{-k_n x}\sin\omega t \\ + i\left\{ \displaystyle\sum_{n=1}^{\infty} a_n \frac{K}{k_n}\frac{\cos k_n(z-h)}{\cos k_n h} e^{-k_n x}\cos\omega t - a_0 \frac{K}{k}\frac{\cosh k(z-h)}{\cosh kh}\sin(kx-\omega t) \right\} \end{array} \right\} \right].$$

$$\therefore \Phi(x,z,t) = -a_0\theta_0\omega \frac{K}{k}\frac{\cosh k(z-h)}{\cosh kh}\cos(kx-\omega t) - \theta_0\omega \sum_{n=1}^{\infty} a_n \frac{K}{k_n}\frac{\cos k_n(z-h)}{\cos k_n h} e^{-k_n x}\sin\omega t. \tag{41}$$

The velocity component in x-direction in the following form

$$u=\frac{\partial\Phi}{\partial x}=a_0\theta_0\omega K\frac{\cosh k(z-h)}{\cosh kh}\sin(kx-\omega t)+\theta_0\omega\sum_{n=1}^{\infty}a_n\ K\frac{\cos k_n(z-h)}{\cos k_n h}e^{-k_n x}\sin\omega t. \tag{42}$$

Now, at the bottom surface,

$$u\big|_{z=h}=\frac{\partial\Phi}{\partial x}\big|_{z=h}=a_0\theta_0\omega K\frac{1}{\cosh kh}\sin(kx-\omega t)+\theta_0\omega\sum_{n=1}^{\infty}a_n\ K\frac{1}{\cos k_n h}e^{-k_n x}\sin\omega t.$$

$$\frac{\partial u}{\partial x}\big|_{z=h}=a_0\theta_0\omega Kk\frac{1}{\cosh kh}\cos(kx-\omega t)-\theta_0\omega\sum_{n=1}^{\infty}a_n\ Kk_n\frac{1}{\cos k_n h}e^{-k_n x}\sin\omega t.$$

and

$$\begin{aligned}u\frac{\partial u}{\partial x}\big|_{z=h}&=\left[a_0\theta_0\omega K\frac{1}{\cosh kh}\sin(kx-\omega t)+\theta_0\omega\sum_{n=1}^{\infty}a_n K\frac{1}{\cos k_n h}e^{-k_n x}\sin\omega t\right]\\&\left[a_0\theta_0\omega Kk\frac{1}{\cosh kh}\cos(kx-\omega t)-\theta_0\omega\sum_{n=1}^{\infty}a_n\ Kk_n\frac{1}{\cos k_n h}e^{-k_n x}\sin\omega t\right]\\&=\theta_0{}^2\omega^2K^2\left[a_0\frac{1}{\cosh kh}\sin(kx-\omega t)+\sum_{n=1}^{\infty}a_n\frac{1}{\cos k_n h}e^{-k_n x}\sin\omega t\right]\\&\left[a_0k\frac{1}{\cosh kh}\cos(kx-\omega t)-\sum_{n=1}^{\infty}a_n\ k_n\frac{1}{\cos k_n h}e^{-k_n x}\sin\omega t\right].\end{aligned}\tag{43}$$

At $t=0$,

$$\begin{aligned}u\frac{\partial u}{\partial x}\big|_{z=h}&=\theta_0{}^2\omega^2K^2\left[a_0\frac{1}{\cosh kh}\sin kx\right]\left[a_0k\frac{1}{\cosh kh}\cos kx\right]\\&=a_0^2\theta_0{}^2\omega^2K^2k\frac{1}{\cosh^2 kh}\sin kx\cos kx.\end{aligned}\tag{44}$$

The governing equation of a stream function for the mass transport of the boundary layer is given by Eq. (11) and the boundary conditions on the free surface, water bottom and the flap surface are summarized as

$$\frac{\partial^2}{\partial z^2}(\Psi(x,0))=\mathrm{Re}\left(\frac{4}{i\omega}\frac{\partial w}{\partial x}\frac{\partial u^*}{\partial x}\right). \tag{45}$$

$\frac{\partial}{\partial z}(\Psi(x,h))=\mathrm{Re}\left(\frac{5-3i}{4i\omega}u\frac{\partial u^*}{\partial x}\right)$, where asterisk $(*)$ is the complex conjugate.

$$=-\frac{3}{4\omega}u\frac{\partial u^*}{\partial x}. \tag{46}$$

and

$$\frac{\partial}{\partial x}\Psi(0,z) = -\operatorname{Re}\left\{\frac{5-3i}{4i\omega} w \frac{\partial w^*}{\partial z}\right\}. \tag{47}$$

Hence Eq. (46) becomes

$$\begin{aligned}\frac{\partial}{\partial z}\left(\Psi(x,h)\right) &= -\frac{3}{4\omega} a_0^2 \theta_0{}^2 \omega^2 K^2 k \frac{1}{\cosh^2 kh} \sin kx \cos kx \\ &= -\frac{3}{4} a_0^2 \theta_0{}^2 \omega K^2 k \frac{1}{\cosh^2 kh} \sin kx \cos kx.\end{aligned} \tag{48}$$

Therefore, Eq. (1) can be written as

$$\begin{aligned}\overline{U_1} &= -\frac{3}{4} a_0^2 \theta_0{}^2 \omega K^2 k \frac{1}{\cosh^2 kh} \sin kx \cos kx \\ &= A \sin kx \cos kx.\end{aligned} \tag{49}$$

where, $A = -\frac{3}{4} a_0^2 \omega k \frac{1}{\cosh^2 kh}$ (50)

Again, the velocity component in z-direction in the following form

$$w = \frac{\partial \Phi}{\partial z} = -a_0 \theta_0 \omega K \frac{\sinh k(z-h)}{\cosh kh} \cos(kx - \omega t) + \theta_0 \omega \sum_{n=1}^{\infty} a_n \, K \frac{\sin k_n (z-h)}{\cos k_n h} e^{-k_n x} \sin \omega t.$$

Therefore, at the flap surface,

$$w\big|_{x=0} = \frac{\partial \Phi}{\partial z}\Big|_{x=0} = -a_0 \theta_0 \omega K \frac{\sinh k(z-h)}{\cosh kh} \cos \omega t + \theta_0 \omega \sum_{n=1}^{\infty} a_n \, K \frac{\sin k_n (z-h)}{\cos k_n h} \sin \omega t.$$

$$\frac{\partial w}{\partial z}\Big|_{x=0} = -a_0 \theta_0 \omega K k \frac{\cosh k(z-h)}{\cosh kh} \cos \omega t + \theta_0 \omega \sum_{n=1}^{\infty} a_n \, K k_n \frac{\cos k_n (z-h)}{\cos k_n h} \sin \omega t.$$

and

$$\begin{aligned} w\frac{\partial w}{\partial z} &= \left[- a_0 \theta_0 \omega K \frac{\sinh k(z-h)}{\cosh kh} \cos \omega t + \theta_0 \omega \sum_{n=1}^{\infty} a_n \, K \frac{\sin k_n (z-h)}{\cos k_n h} \sin \omega t\right] \\ &\left[- a_0 \theta_0 \omega K k \frac{\cosh k(z-h)}{\cosh kh} \cos \omega t + \theta_0 \omega \sum_{n=1}^{\infty} a_n \, K k_n \frac{\cos k_n (z-h)}{\cos k_n h} \sin \omega t\right]\end{aligned}$$

$$= \theta_0{}^2 \omega^2 K^2 \left\{ a_0 \frac{\sinh k(z-h)}{\cosh kh} \cos \omega t - \sum_{n=1}^{\infty} a_n \frac{\sin k_n (z-h)}{\cos k_n h} \sin \omega t \right\}$$
$$\left\{ a_0 k \frac{\cosh k(z-h)}{\cosh kh} \cos \omega t - \sum_{n=1}^{\infty} a_n\, k_n \frac{\cos k_n (z-h)}{\cos k_n h} \sin \omega t \right\} \tag{51}$$

At $t = 0$

$$w \frac{\partial w}{\partial z} = \theta_0{}^2 \omega^2 K^2 \left\{ a_0 \frac{\sinh k(z-h)}{\cosh kh} \right\} \left\{ a_0 k \frac{\cosh k(z-h)}{\cosh kh} \right\}$$
$$= \theta_0{}^2 \omega^2 K^2 a_0^2 k \frac{1}{\cosh^2 kh} \sinh k(z-h) \cosh k(z-h). \tag{52}$$

Also Eq. (46) can be written as

$$\therefore \frac{\partial}{\partial x} \Psi(0, z) = -\mathrm{Re}\left\{ \frac{5-3i}{4i\omega} w \frac{\partial w^*}{\partial z} \right\}$$

$$= \frac{3}{4\omega} w \frac{\partial w^*}{\partial z}$$

$$= \frac{3}{4} a_0^2 \theta_0{}^2 \omega K^2 k \frac{1}{\cosh^2 kh} \sinh k(z-h) \cosh k(z-h). \tag{53}$$

Therefore, Eq (2) can be written as

$$\overline{W_1} = -\frac{\partial \Psi(0, z)}{\partial x}$$
$$= -\frac{3}{4} a_0^2 \theta_0{}^2 \omega K^2 k \frac{1}{\cosh^2 kh} \sinh k(z-h) \cosh k(z-h) \tag{54}$$
$$= A \sinh k(z-h) \cosh k(z-h).$$

From Eqs. (49) and (54), mass transport velocity has been formulated.

2.4. Time Average of Horizontal Velocity Component

Time average of horizontal velocity *u* can be expressed as

$$\bar{u}(x, 0, t) = \frac{1}{T} \int_0^T u(x, 0, t) dt. \tag{55}$$

Hence Eq. (42) can be written as

$$\bar{u}(x, z, t) = \frac{1}{T} \int_0^T \left[a_0 \theta_0 \omega K \sin(kx - \omega t) + \theta_0 \omega \sum_{n=1}^{\infty} a_n\, K e^{-k_n x} \sin \omega t \right] dt$$

$$= \frac{1}{T} \left[a_0 \theta_0 \omega^2 K \cos(kx - \omega t) - \theta_0 \omega^2 \sum_{n=1}^{\infty} a_n\, K e^{-k_n x} \cos \omega t \right]_0^T = 0.$$

Using eigen function expansion, velocity potential has been derived. Then we see that time average of horizontal velocity component is zero, which is same as Dean and Dalrymple [15], who derived some nonlinear properties for water waves of small amplitude.

3. Conclusion

Mass transport velocity in two dimensional wave tank has been studied. Vorticity equation from incompressible Navier-Stokes equation is solved using boundary layer theory and boundary conditions on the free surface, water bottom and the flap surface. Generalising the velocity potential in terms of eigen function expansion, we obtain the velocity components which are related to stream function. Using the given boundary conditions, mass transport velocity has been formulated. Then we see time average of horizontal velocity component is zero at mean water level.

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